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TEACHING CRITICAL AND CREATIVE THINKING SKILLS THROUGH PROBLEM-SOLVING IN HIGH SCHOOL MATHEMATICS CLASSES

by

DANIEL L. ALBERT
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SYNTHESIS*
MASTER OF ARTS
CRITICAL AND CREATIVE THINKING
UNIVERSITY OF MASSACHUSETTS BOSTON

May 2016

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Abstract: I am a high school mathematics teacher who wants students to develop the thought process that allows them to uncover and use the patterns that are the heart of mathematics. Misconceptions belittle the subject, however, and disrupt the progress of my high school students because it disguises the purpose of studying math. My purpose in joining the Critical and Creative Thinking program at the University of Massachusetts Boston was to learn about the skills and methodologies that assist in the teaching and learning of thinking critically and creatively. This paper shows the progress and evolution of my capstone project—a set of lessons that explicitly teach useful thinking skills in a problem-solving context. This project combines aspects of my learning from various courses, personal experiences, and research. It follows me through developing and implementing a plan to create and teach the curriculum with special emphasis on reflection and learning from the experience in order to improve the lessons.

[*] The Synthesis can take a variety of forms, from a position paper to curriculum or professional development workshop to an original contribution in the creative arts or writing. The expectation is that students use their Synthesis to show how they have integrated knowledge, tools, experience, and support gained in the program so as to prepare themselves to be constructive, reflective agents of change in work, education, social movements, science, creative arts, or other endeavors.

Special Thanks

I would like to express my gratitude and acknowledge the contributions of the many people who made this project possible.

I would like to add a special thank you to:

Jessica Collins, for editing, answering questions, and generally helping make what I wanted to say understandable.

Paul Goldenberg, for allowing me to interview you, for giving wonderful advice, and for generally being an awesome person who finds the time to share your love for math and teaching.

Kathy Foulser, for allowing me to use many of the problems we did in classes and for modeling how the learning experience can go.

Peter Taylor, for working with and guiding me from the beginning of the CCT program to the end.

Jeremy Szteiter, for the helpful guidance, thorough feedback, and important comments as I progressed through the synthesis project.

My colleagues and peers in the CCT Program, for all of the classes, discussions, work, and experiences we shared.

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Introduction

Background

In a sermon, a Rabbi once said that one's personality is how one behaves around others while the character is what is engraved upon one's soul. If that is the case, then being a teacher is part of my character. In every aspect of my life, I endeavor to help others understand and strive to share my knowledge. Although my current profession is that of a high school math teacher, and I know that it will not last forever, I cannot imagine a time in my life when I will not be a teacher of some kind. When it came time to choose a topic for my synthesis project, it seemed clear to me that it would pertain to my teaching and to my ability to help students see the world as I do.

I love learning. Every day I search for opportunities to discover new things. In order to do this, I need to use different strategies and mindsets to unravel the mysteries of our world. This is why I have chosen to research and look for ways to teach these methods to my students, so that they, too, can find the joys in life's puzzles and patterns. Patterns are found throughout the world we live in, and are also intricately connected to mathematics. I believe that mathematics is the study of recognizing, modeling, and using patterns. Since I am a teacher, I want to help my students understand these things and be able to use them.

When I look back at my math classes throughout my schooling experience, I am struck by how many opportunities I had to expand upon what I had been taught and to stretch my understanding of things further than before. I was given chances to see new situations and delve into them with the hope of finding a mathematical pattern that would explain it. These experiences gave me a love for a subject that can be found in practically everything. They

allowed me to develop a sense that nothing was beyond comprehension, and that patterns can be found if we take the time and know some strategies to look for them.

Unfortunately, I find that many of my students do not see math as I do. They do not see how it applies to anything beyond the classroom. They do not recognize that it is through math that we describe the patterns of our world. Many students have complained that math does not make sense, that it is not logical, and that it is not interesting. It pains me to hear students express these disheartening ideas when I know the opposite to be true. This project is designed to address these deficiencies in the understanding of what mathematics is. It is a series of lessons that aim to teach and develop the skills and disposition to use critical and creative thinking in a problem-solving context, so that students are prepared to encounter new and unknown problems in a variety of situations.

Identifying the Problem

I have seen my students presented with something new and freeze, saying anything from "I don't know how to do this," to "I'm just going to wait until the teacher goes over it." These tendencies prevent students from engaging with the situation, and preclude the possibility that they will learn, because they never even begin. Once students do begin, I have seen too many give up after a single failed attempt. Many of my students lack the persistence required to unravel any situation complex enough to be worth understanding. When my students become frustrated, I often tell them "if you can solve a puzzle on your first try, it is not much of a puzzle." In that same vein, I have found that students too quickly jump to asking for help. There are times when asking a question is entirely appropriate, but there are also times when asking replacing thinking. I want my students to understand the difference, and to have strategies that allow them to successfully work on problems without needing to ask someone else. These three

abilities are what separate the students I currently see struggling from ones who will be able to recognize and understand the patterns that exist both within math class and without.

In order to recognize the mathematical patterns in any given situation, students must be able to do three things: they need to be able to (1) begin problems and approach situations, whether or not they have seen that specific kind before, (2) persist through challenge and failure to arrive at a conclusion in which they are confident, and (3) trust and rely on their own understanding and thought process to reach that conclusion. I believe that these abilities are worth addressing for a multitude of reasons. The foremost of which is that these abilities will allow students to make use of the opportunities they are given to learn and understand mathematics. If they are able to do these things, then they will be better prepared and able to get more out of their math classes, current and future. If they are able to do that, then they might just see mathematics as the interesting and compelling subject that I do.

Additionally, these skills should prove useful in other areas. Students who are able to approach and persist in working on problems on their own make better use of their class time. This also allows teachers to spend their time in class more efficiently, working with the students who cannot approach the content or who need assistance to do so. These skills should also prove useful on tests, whether standardized or otherwise. Students must be prepared to face novel situations on assessments, and they must be able to proceed without the help of a teacher. Finally, students with these abilities should be better able to face new circumstances outside of school. People cannot be trained for every possible situation, but these skills should help when faced with the unexpected.

Why Me

It is unfortunate that the existing curriculum does not address these issues, because these skills will assist the students in addressing not only the mathematics materials they are taught,

but also those of other subjects, as well as a variety of situations that they will encounter in their lives post-high school. I am motivated and determined to address this because I truly believe that they are important skills for my students to know. Additionally, I have the background, experience, and predisposition to serve as model and teacher for these skills. I enjoy thinking and problem solving. Completing a task I already understand can become tedious, but encountering novel situations presents a challenge, and provides an opportunity to learn. I want to share that love of and desire for new scenarios with my students. I also have the training I have received as part of the Critical and Creative Thinking program and the University of Massachusetts Boston. This program has guided me to this point, and has given me the opportunity to learn and experience what I need to complete my goals. Finally, my teaching method already includes practices that encourage students to develop and improve upon the abilities I focus on in this curriculum. This gives me the experience to know what works, as well as makes the implementation of this curriculum easier for both my students and myself.

Unfortunately, I do not think it will be an easy process regardless of my qualifications. From my perspective as a teacher, I see students who go through their entire primary and secondary school careers without having those experiences, without having the chance to tackle a problem they have never seen before. As such, they are not developing the skills needed to do so. When these students run into such a problem, many will stop and wait until they are told how to solve it. It appears as though they want a method or an answer without having to put in the work to find it on their own. I believe that this happens for multiple reasons. One reason is that many teachers are overly quick to jump in and provide a path to a solution, resulting in students becoming used to this assistance. I cannot pinpoint the reason teachers do this. I can say, however, that I have experienced a culture where such answers are expected, both by students

and parents. I can also say that it is often easier for teachers to simply give an answer than to come up with an appropriate response that forces the student to find the answer on their own. This reliance on outside input makes it less likely that students will develop the skills, strategies, and tendencies that allow them to tackle the problems on their own. Another reason is that there is some pressure on teachers to only assign questions (particularly on summative assessments) that match fairly exactly what students have previously been taught. When they are presented with these problems, they appear to be more inclined to simply mimic what they've been shown before, without making any effort to understand why the process works the way it does. This is truly an atrocity for these students, as well as for the world they will inherit.

This is why I am developing an add-on curriculum that focuses on critical and creative thinking skills through problem-solving situations. This curriculum is a set of additional lessons meant to complement the existing curriculum rather than replace any part of it. I plan to implement this curriculum into my sophomore geometry classes. These sections are of the lowest academic level that I teach, and some of the most challenging general education classes in the school. Many of the students in these classes have not had success in their past math classes, and an unfortunately large number lack the motivation (and thus do not put in the effort) to do well in school. Additionally, I see the greatest possible benefit for these students. Based on my experience, they are least likely to see math as worthwhile, to make attempts at problems they do not know how to do, and are prone to either ask questions too easily or never ask for the help that would allow them to succeed.

Gathering Information

Research Goals

In order to put together a curriculum of lessons to teach the skills of critical and creative thinking, I needed information in several areas. Many of the specific skills that I intend to teach come from my experiences in graduate courses, both through the University of Massachusetts Boston, and through professional development at my school. My goals for the research included finding methods of teaching or practices to incorporate into the lessons that improve the lessons and the curriculum as a whole, additional ideas for topics or lessons to be included, and sought advice on the construction of a curriculum, which I had never done before.

What Classes Have Taught Me

To date, I have taken multiple classes that have given me important information to help me accomplish my goal. Some of these classes have been part of the Critical and Creative Thinking Program at the University of Massachusetts, Boston, and others have been provided by various Colleges through the school where I teach. Several of these classes have spoken of the importance of a "Growth Mindset," as described by Carol Dweck (Dweck, 2007). The idea of a Growth Mindset is that people can improve on just about any task with practice and effort. People are neither naturally gifted nor limited in tasks to the point that hard work cannot help them improve. Even in my limited teaching experience, I have found that the idea that intelligence is not fixed to both conflict with previously held student beliefs and be very helpful to students once they have accepted it. Many students have been stuck in a mental loop wherein they think that they are "bad at math", so they do not put in much effort. This usually leads to the conclusion of the self-fulfilling prophecy. By introducing students to the research and concepts of Dweck, many are able to break out of their cycle of failure.

I have also taken a class on the implementation of one part of the Common Core Math Standards. This aspect of the standards, known as the eight Standards for Mathematical Practice (SMP), is a collection of processes that mathematicians do on a regular basis that permeates almost every part of our professional practice, and includes critical thinking skills, such as recognizing patterns and critiquing the reasoning of others (National Governors Association Center for Best Practices, 2010). I found a significant amount of overlap between these standards and the thinking and problem-solving skills that I believe are important for my students to gain from my class. Additionally, the SMPs give me an important entry point into what I already teach. This is important for when I need to justify my use of class time, because I can use the SMPs as a bridge between the existing curriculum and my new one.

The classes I have taken through the University of Massachusetts Boston have focused on the very critical and creative thinking skills that gave the degree program its name. In each of my classes for this program, I have applied the course material to my teaching in the hopes of assisting my students in developing their critical and creative thinking skills, and have included what I learned in these classes in my curriculum. These classes have shown me some ways in which creativity and critical thinking skills can be fostered, including behavioral, educational, and environmental factors. Some of these ideas I already held, such as listening intently to students as they explain their reasoning, even if I know it to be flawed, and then helping them discover the good pieces along with the bad. Others have been more foreign, but still implementable, such as designating a space in my classroom for puzzles that students can access when they have extra time throughout the school year. These puzzles, which have been a huge success with a significant portion of my students, allow them to explore spatial, logical, and creative impulses on their own terms.

Other Concepts to Include

In my search, I found a number of sources that either support my plan or help guide me as I shape it. Rowlett (2011) wrote about how failure is a key part to the learning experience. Students are often afraid of making mistakes, he claims, and this fear holds them back. When students are unwilling to make an attempt, they prevent themselves from a crucial learning experience. Teachers should encourage students to embrace mistakes, and use them to everyone's advantage. Rowlett says, "when a student answers a question incorrectly, the teacher should use that mistake as an opportunity to guide the student towards the correct answer" (Rowlett, 2011, p. 38). Mistakes not only point out an error that can be avoided in the future (both by the student who made it and his/her classmates), but they give everyone the opportunity to test their understanding of the material, and gain practice with understanding and critiquing the arguments of others, which is also one of the Standards for Mathematical Practice. Rowlett goes further and makes the following statement: "To pose problems and foster creativity, teachers must encourage students to work through their failures and correct their thinking. Teachers should employ a rubric whereby students are applauded for creativity, risk taking, and collaboration" (2011, p. 38). This statement not only gives advice to the teachers, but also gives a tool to be used in the process.

Two other authors I found explicitly reference and build upon the work of Carol Dweck on Growth Mindsets. Blad (2015) encourages teachers to praise students for their effort, at least initially, in an attempt to build up their confidence and belief that hard work allows them to improve. Additionally, Blad recommends removing comments that attribute success to innate intelligence from the vocabulary of both teachers and students. Statements such as "great job, you're so smart!" should be replaced with "great job, I can see you worked really hard on that!" Blad goes on to recommend some classroom activities that play into the aspects of a Growth

Mindset (2015). Similarly, Sparks (2013) points out how such changes have already been implemented in many schools. She uses examples from New Orleans schools to show not only how classes and teachers can be modified to enforce a Growth Mindset, but also how the way a school is run and how classes are chosen can, too.

I was also able to find some work from the National Council of Teachers of Mathematics (NCTM), one of the biggest advocates for mathematics teachers and education in this country. In addition to creating and distributing resources for math teachers and schools, NCTM makes recommendations that politicians (supposedly) take into consideration when forming education policy. NCTM's stance has been that "Problem solving plays an important role in mathematics and should have a prominent role in the mathematics education of K-12 students" (Cai & Lester, 2010). The authors go on to give a definition of problem solving, some criteria to help teachers choose what counts as a worthwhile problem, and some examples of how ordinary problems can be adjusted to create a more rich learning experience for students.

The NCTM also released information on discussion as an educational method. Cirillo states that "discussion provides an alternative to recitation. Within discussion, assessing students' subject-matter knowledge is not necessarily the primary and sole objective. In addition, teachers are interested in helping their students to develop understandings" (2013, p. 1). She is making the case for including more discussion in math classes to better understand the reasoning students have and why they get the answers they do. She lists several benefits to discussion, such as increasing student learning, increasing motivation, and supporting teacher understanding of student thoughts (Cirillo, 2013).

Maida and Maida (2011) have a different approach. They encourage the use of games in class to help with student motivation, as well as giving opportunities for problem-solving skills

in a different context. Their article focuses on a game called Blokus that involves placing differently shaped tiles strategically to expand and protect territory, so that you can place more tiles in your future turns. The game is very geometric, as well as logical. They propose using it to force students to both plan and explain strategies, including why they would choose one move over another. Maida and Maida also include example lesson components and questions to use the game to prompt student thinking. (2011). An example of a situation where students can explain their reasoning is given in figure 2-1.

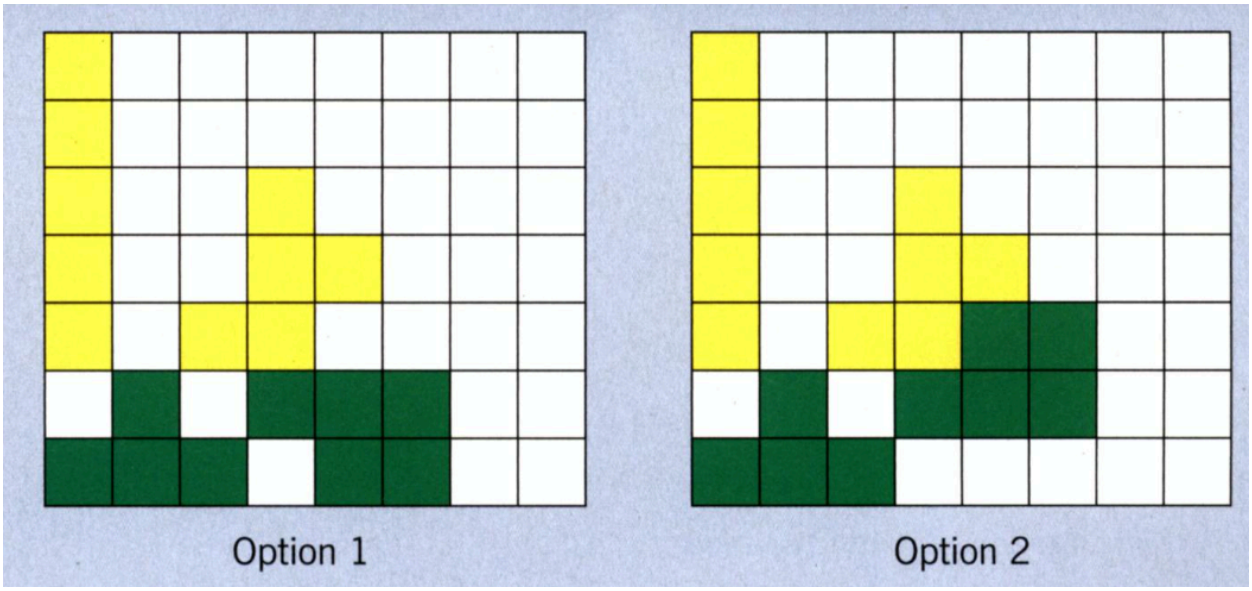

Figure 2-1. Maida and Maida give the situation shown to the left as an example of the opportunity for logical reasoning and explanations in the game of Blokus. Students could be given the two positions and instructed to give a reason why they would choose to play the green piece in one place or the other.

I also learned an important lesson from Ewald (2012) as she attempted to make lessons in communication more relevant to today's students through the use of Twitter. Ewald created and implemented a project that used the formatting of the social media site while students summarized geometric concepts. Ewald hoped that, although students wrote their summaries on paper, the assignment would replicate the feel of the social media site, producing a heightened sense of engagement and enjoyment. As it turns out, the majority of the students who participated did not respond positively to the project (Ewald, 2012). Ewald's negative results

remind me that first attempts are not always effective. That being said, it is certainly possible to learn from negative experiences and revise them based on the feedback and experience gained.

Krulik and Rudnick (1994) present another take on a creative classroom. In their article, they described a class wherein student questions in the course of problem solving are discussed and determined by other students. For example, in a problem referencing rectangles, one student asked if squares count as rectangles. The teacher could easily have given the correct mathematical answer (yes), but the teacher allowed and prompted other students to discuss and respond. In this way, the rest of the class also became interested and invested in the question. Additionally, they were able to reason together and come up with an answer that guided the rest of the problem-solving activity (Krulik & Rudnick, 1994).

Creativity is also a key aspect of problem solving. Davis (1969) claims that there are three main creative traits to focus on: (1) an attitude that enjoys creative solutions, (2) cognitive abilities that allow for mental adaptations and/or manipulations, and (3) techniques for the creation of new combinations of ideas. Davis also claims that adolescence is an important time for the development of creativity, an idea supported by Dai and Shen (2008). The latter goes on to offer some advice for fostering creativity, including encouraging exploration and inquiry, as well as the taking of reasonable risks. This matches up well with the work of Rowlett (2011). Risks can lead to failure, which is as important for creativity as it is for learning in general.

Constructing a Curriculum

In order to get the point of view of someone who has designed mathematics curricula for decades, I interviewed Paul Goldenberg, a friend of mine who works for the Education Development Center. I chose to speak to him to gain insight into the process and lessons learned from actually creating and improving curricula.

The largest point that Goldenberg made in his responses to my questions is that the most critical component of creating a curriculum is not the theory. It is possible to design something that looks great on paper and *should* work perfectly, but does not. In order to design something that will work, I need to base my ideas on information gathered from watching students as they interact with and try out ideas. He encouraged me to work with students, to watch them play with ideas, and to use these observations to help craft my curriculum.

Goldenberg also passed along some other helpful pieces of information. He told me that curriculum design takes a lot of work, effort, and revision, suggesting that I would likely have to go through multiple implementations to get to something I felt confident about. To balance this point out, however, he also said that just because something is a revision on an older attempt does not mean it will be an improvement.

He also conveyed one more piece of wisdom in response to a concern I had about my project. I had been concerned that my students may not benefit from having me teach and emphasize these skills, and then going onto another teacher the following year that does not. I worried that the lack of consistency would send mixed messages that detract from the lessons I hope my students learn. He assured me that my students would not be worse off for having a positive educational experience with me. His wife, Cindy Carter, who is a practicing math teacher, seconded his assurance. She has often felt alone in her department in terms of the teaching methodology she uses. Despite this, Carter is confident that her students benefit from the experiences they have in her class, even if they are not supported in future years. In fact, her students report that they have been better able to learn in their other math classes because of her teaching style (personal communication, November 14, 2015).

It is my distinct hope that my students will be better off for the time and effort we put in together on this project. Part of my plan is to discuss the results with my supervisor and the head of my department. If the results are promising enough, either these lessons or an offshoot of them might be applied to the rest of the department. That being said, the skills that my students learn from this curriculum can certainly be applied to their later mathematics courses with or without the promptings of their future teachers. I expect that some of my students will, in fact, use these skills and strategies as they encounter new situations in their future courses and beyond, but I also expect that others will not. If I am able to develop and follow through on an appropriate plan, however, I hope to make it so that more students fall into the first of those two groups.

Action Plan

Action Plan Purpose

A curriculum does not spring fully-grown from the head of some Zeus, so I set forth to develop a plan that would guide me through the development, implementation, and revision of mine. I began by contacting the people whose support I would need, and who could assist me in the construction of the curriculum, which was the second step. I then put together a plan for an initial implementation in fall 2015, with the aim of collecting data that would help me revise and improve upon my first edition.

Stakeholder Support

In order to complete a project like this, I needed the help and support of several people. The first person that I convinced to become a constituent was my direct supervisor, Gale Bock, the head of curriculum and instruction at my school. She has been very supportive of the entire department as we try to better our instruction. We discussed the situation, and she gave me advice regarding how I can better design and evaluate my project. She also gave me permission

to use class time on this endeavor. Although projects like this are expected of teachers at my school (as a part of our annual evaluation process), I expected it to be larger and to take up more class time than most, which has the potential to have a larger impact on my classes (for better or for worse). Mrs. Bock also agreed to help me along the way as I gather formative data and continue to tweak the process. While her advice is valuable, the biggest reason I went to her first is that she needs to approve the unit plans and overall curriculum that is taught in my classroom. While this usually happens as I reach each new unit during the year, a curriculum change of this magnitude called for pre-approval. I felt that attempting this project without getting her approval would be inconsistent from an educational standpoint, and possibly dangerous for my continued employment.

Another person whose support I obtained is Kathy Foulser. Ms. Foulser taught one of the graduate classes I took through my school, and has been the source of many of the experiences and problem situations that inspired specific lessons in my curriculum. I also spoke to Teresa Katuska, the head of the math department. She has been working on ways to improve our department's methods of bettering student achievement, and has had a significant amount of useful advice. We are also working together to spread it to more of the department, pending completion of my initial attempts.

As my project has progressed, I have felt it more and more important to think of my students as constituents as well. While they are certainly participants who benefit from what I do, I had not originally thought of them as being contributors to it. I realized, however, that this project cannot succeed if my students are not engaged and invested in it. I need to ensure that they are interested and put in the required effort to make it work. This will be a difficult task, and I highly doubt that I will be able to convince *all* of my students to buy in, but I will actively work

towards doing so with every implementation, and will continue to seek the assistance of others to make it happen.

Development of the Curriculum

At this point in my project, I had to put serious thought and time into deciding how I was going to measure the growth that my constituents are interested in seeing. I started by deciding what specific things I was trying to change and how I could measure them. Since I want to focus on the students' ability to approach problems they have never seen before, I thought about either trying to measure their belief in their abilities or look at actual problems they have attempted to solve. Seeing as it would be very challenging to assess the individual attempts that an entire classroom full of students are making simultaneously, I settled on using a questionnaire to gauge how each student feels about their abilities and confidence in these areas.

I designed a survey (Appendix B), part of which asks questions about how strongly they agree with statements, and part of which asks for students to complete sentence starters that I hope will allow me to see what is going on in their heads. The first section contains ten questions, only four of which are what I would consider necessary for my intentions. The rest are mostly meant to see if there are correlations between what I am trying to change and other opinions students hold, as well as partially hiding the true aim of my survey so that students do not simply respond with what they think I want to hear. I know from experience that many of my students respond to my questions during class as they believe I would want them to, so I saw this as necessary. The second section is similar in its goals. By looking at the responses to these questions before lesson 1 and after lesson 11 (approximately 3 months later), I hope to gather evidence to support whether or not I have made a difference in my attempts to do so.

I then looked to the graduate classes I have taken to see what skills are important to critical and creative thinking that would be applicable to mathematics problems (see Appendix

A). With so many topics covered in my classes, I chose skills that I felt were applicable to problem solving, particularly in a mathematics context. It helps that, throughout my courses, I was already looking at them with the purpose of finding and teaching skills to my students. I also looked at what sample problems I have done in these classes that could be adapted for my students, particularly if these problems were already paired with a skill or lesson. By doing this, I found a set of problems that could be used to teach these skills as well as a handful of mathematical concepts. I put these together to form the basis for my curriculum. These lessons focus on a combination of problem-solving strategies and mathematical concepts, but the real emphasis is on the former. Those skills, not the mathematical connections, are what I hope students will bring from one lesson to the next as they build their abilities in this curriculum.

For the structure of the lessons themselves, I settled on a basic process that would be modified for the individual problem and/or skill to be taught. Most lessons would begin with a few minutes for students to read the problem and ask clarifying questions to ensure their understanding thereof. There would then be a period of time given so that they could work with either manipulatives or paper and pencil to respond to the directions and the questions towards discovering the answer or pattern of the problem. Time would then be taken at the end for students to discuss their methods as well as their results, emphasizing the myriad ways to approaching and solving each problem. The teacher would then lead a summary of the skill the problem was meant to focus on as well as the mathematical topic(s) found in the problem, if any.

The Implementation Process

There were two primary stages to the implementation of my curriculum. The first was a complete run-through of the curriculum, including data gathering and teaching the lessons. The second step was to reassess and revise the curriculum based on the information gathered in the

process. Based on my results, this was a series of minor tweaks to some lessons, as well as a more major overhaul of the data gathering process.

The initial round of implementation was put into motion at the start of the 2015-2016 school year. I had planned to use some time every week to work towards this goal. I did this to make use of a relatively small amount of class time (roughly half of a class per week) for a preliminary four-month period, with the possibility of being expanded to the entire school year. I collected data on how students viewed their own capability to approach new problems (via the questionnaire described above) at the beginning and end of the planned curriculum. The class time was most commonly Friday afternoons, but that was adjusted depending on the specifics of the week, as well as any assessments or special schedules that took place.

After the initial survey, the first problem I gave was very approachable. It was something they had likely never seen the specifics of before, but something whose level of mathematics was well within their grasp and whose logical requirements were minimal. The problem in question can be found in Appendix F. The goal of this was to introduce the idea that they are capable of solving problems they had not been taught to handle. The burden on me was to create the appropriate problem, to help them understand it, mostly through modeling, and then to bring it all together at the end. Following the initial problem, I taught some mini lessons on ways to approach problems that are new to them. These lessons focused on skills and processes which can help with both the understanding of a problem and the act of getting started, such as modeling with mathematics and using manipulatives to better understand a problem. These, along with other ideas I introduced, came from the various courses I have taken, including one on the implementation of the eight Mathematical Practices described in the Common Core State

Standards. I reasoned that once they have a firm footing for getting started, the rest of the task becomes much easier.

After that, things settled into something of a routine with the weekly sessions being mostly comprised of a new problem for them to try and approach. After students had some time to work on each one, we reconvened as a class to discuss what methods they used on said problem, why they chose that method, and what they learned. The emphasis, particularly at first, was on the process of understanding the problem and attempting to find a solution rather than on the solution itself. Over time, I found that more and more students were willing and able to join into the conversation. I also hoped that they would see the utility of the skills taught in the mini-lessons as well as seeing some different methods. It also turned into an ideal time to teach some mathematical patterns or operations that might not otherwise have come up in their high school career.

In addition to these specific lessons, I also incorporated some more structural changes into my classroom. I had puzzles available to students who finished their work early in class, particularly quizzes or tests. These puzzles were popular with a large segment of my students, and helped them develop spatial reasoning as well as creative and critical thinking skills. I also used these puzzles as a classroom activity several times.

I made a point to defend students' ability to freely express thoughts and ideas related to the different methods of problem solving and mathematical content. Unfortunately, many students were scared of making comments or answering questions, because when they did in past classes, other students mocked them for what were perceived to be stupid statements or questions. I wanted to make sure that my classroom was an intellectually safe space. Poking fun at someone because they thought something or because they had not learned something before

was in no way tolerated. By implementing this portion, I hoped to ensure that students could express their ideas, admit their lack of knowledge, and learn from the experience without fear of facing negative social consequences.

After giving the questionnaires each time, I collected the data into a spreadsheet where I was able to track any changes. For the first ten questions, I converted the responses on the scale of 'disagree' to 'agree' into numbers from 1 to 4, and computed the averages and frequency of responses for each question. As stated above, I mostly focused on certain questions, but I was interested to see how the student responses changed on all of them. Based on the data I gathered this year, I will create goals for how much I would like to see the responses increase in future implementations. These goals would most likely be an increase of somewhere between a quarter and a half of a response on average for the classes. I also compared the responses to the sentence completions to see if the number of 'helpless' responses (i.e. students who say that, when faced with a problem they've never seen before, wait for a teacher to help them) decreased.

Based on the data that I gathered, I brought the results to Mrs. Bock and Mrs. Katuska, my supervisor and the head of my department, and we discussed what it could tell us about our efforts as a department and as a school. Since the data was not yet where we wanted it to be, I made modifications to my project in order to possibly try again next year with the goal of improved results.

In addition to this specific evidence, I also looked to see how my students enjoyed the experience. Part of my reason for doing this project has been student motivation—I want students to be active participants in my class and to (ideally) look forward to school and math class. To this end, I both directly asked students and observed their behaviors during and attitudes towards class. I was able to compare this to three years of anecdotal evidence I used as

a basis for how much students currently enjoy my class. If I can adjust my project to increase student enjoyment (without cutting out the content), I will certainly do that.

Results and Analysis

Summary of Data Gathered

At the beginning of the school year, prior to any of my lessons being taught, and after the conclusion of the final planned lesson, I gave my three classes of sophomores, totaling around 40 students, a questionnaire with the intent of gathering information on the past experiences of the students as well as getting a sense of how they view their own abilities. Table 4-1 shows the number of students who gave each response to the questions prior to lesson 1. An updated version of the questionnaire can be found as part of the curriculum in Appendix B.

Question		Disagree	Somewhat Disagree	Somewhat Agree	Agree
1	I like math class.	3	9	18	10
2	I enjoy learning new things in math.	3	6	19	12
3	I always know how to approach new problems.	4	13	16	6
4	I see math as useful to my future.	0	3	22	13
5	I have learned things in my past math classes.	0	4	12	24
6	When I start a problem I've never seen before, I try at least one thing to solve it.	0	2	25	13
7	I know several problem-solving strategies.	4	9	19	8
8	I have been taught problem-solving skills (ways to solve more than one kind of problem) in school.	0	5	22	13
9	I want to do well in school.	0	0	7	33
10	If I don't solve a problem on my first attempt, I will keep trying until I can get it.	0	7	26	6

Table 4-1. Number of responses per question on the initial questionnaire, given before any lessons were taught.

From this information, I gather that most of my students do not actively dislike math. Many of them, as indicated in questions 1 and 2, actually enjoy both math class and learning new things in it. This bodes well for my purposes, because the skills I emphasize and the

methodology I employ center around learning and discovering. Similarly, a great majority see math as useful to their future and have a mental connection between math and learning things, as seen in questions 4 and 5. These both help with student motivation and (I expect) effort. Number 9 also gives me hope that my students will be motivated to participate and engage with the curriculum. Number 3, actually being the most negative in student responses, shows me that students have a lot to learn from my lessons.

Of particular interest to my student learning goals are numbers 6, 7, 8, and 10. These were the questions that referred to problem solving. Surprisingly to me, only number 7 shows signs of students who are not confident in the areas of problem solving. Students begin this curriculum confident that they have a way to approach problems, and the persistence needed to keep at it until a solution is found. The relatively high responses to 6, 8, and 10 make me question several things, because they are higher than I expected. It seems possible to me that students overestimate their own abilities, particularly because my observations of students seem to contradict their confidence. It is also possible that students do not fully understand what I mean by those questions. A third possibility, which I must entertain, is that my reading of the situation is wrong, and these lessons are unnecessary and unhelpful.

Table 4-2 shows the responses to the same questionnaire given in December, after the lessons had been taught. The two data sets are rather similar, which ran contrary to my expectations. Average response to each question changed no more than .25, which amounts to approximately 10 students changing their response by one level. In addition to this effect being low in magnitude, several of the questions showed a decrease in average response. Questions 3, 5, and 7 showed a negative change between the two surveys. While number 3 could conceivably have decreased by virtue of students having a wider perspective on problem solving, number 5

should not have decreased because it refers to past years of math classes, and as a result, should not have changed at all between September and December. Although the other questions did see an increase in that time frame, the magnitude of the change and the negative responses make me question their validity.

	Question	Disagree	Somewhat Disagree	Somewhat Agree	Agree
1	I like math class.	2	6	22	8
2	I enjoy learning new things in math.	1	6	23	9
3	I always know how to approach new problems.	1	17	18	3
4	I see math as useful to my future.	1	2	15	19
5	I have learned things in my past math classes.	0	5	11	23
6	When I start a problem I've never seen before, I try at least one thing to solve it.	1	2	20	16
7	I know several problem-solving strategies.	7	5	17	10
8	I have been taught problem-solving skills (ways to solve more than one kind of problem) in school.	1	1	16	20
9	I want to do well in school.	0	0	5	33
10	If I don't solve a problem on my first attempt, I will keep trying until I can get it.	1	8	19	11

Table 4-2. Number of responses per question on the final questionnaire, given after all lessons had been taught.

Instead, I see this as having one of several causes, similar to my responses to the initial set of data. It is possible that students still do not have a firm grasp on their own abilities, either at the beginning or the end, possibly due to a lack of appropriate metacognitive skills. It is also possible that students did not know what I meant by those questions. For instance, students may not have understood what I meant by “problem-solving strategies” in the beginning, and still may not have had a firm grasp on it at the end. A third possible explanation is that my lessons were ineffective.

For the short answer questions, I created categories of responses that I expected and felt would encompass the majority of student answers and counted how many responses fell into

each category. Table 4-3 shows this information, which allowed me to compare the data similarly to how I compared the above data.

Answer Choices	Question 2: When I start a problem I've never seen before, I…		Question 3: When I don't get a problem on my first try, I…	
	September	December	September	December
Try something	25	18	17	22
Ask	13	10	19	12
Wait	1	3	7	4
Use strategies	8	12	0	3
Other	0	3	8	3

Table 4-3. Short answer questions, divided by general category of response, to questions 2 and 3

First, some responses counted for more than one category. For instance, a response to the third question of "try one more time, then ask for help" would be counted as a 'try once more' and an 'ask for help'. Looking at the changes in question 2, I notice that the answers became more specific. The number who responded with some variant of 'try something' or 'ask for help' decreased, while the number who mentioned specific strategies or something different enough not to fall into one of the expected categories increased. I see this as a positive change, because students are better able to articulate and recognize what it is that they do to start a problem.

The responses for the final question are also encouraging. The number of people who said that they would 'keep trying' increased, which shows a probable increase in persistence amongst the students. While asking for help can be a good thing, I believe the thought processes of those who ask for help in place of trying to figure it out themselves are not the ideal, and I hoped this new form of learning would change that. While it is certainly better than giving up or waiting, I am pleased to see that the number of people who responded with 'ask for help' decreased, and presumably gave another positive response instead. Naturally, the increase in people who mentioned specific strategies is also interpreted to be a good thing, though I am disappointed that it remains relatively low.

In addition to these questions, I asked my students a couple of other questions aimed explicitly at gauging their enjoyment of the process, and on how I could improve it in the future. From this, I learned that the majority of students did enjoy the lessons, and only 5 of the 40 questioned had neutral or negative opinions on the experience. Similar numbers of responses indicated that students feel that they learned from these lessons and are better prepared for similar problems in the future. Given that one of my goals was to have students enjoy the experience in hopes of demonstrating that math can be interesting and enjoyable, I found these numbers to be very encouraging.

Summary of my Reflections

Following each lesson I taught, I wrote a reflection about the experience. I set aside this time to get my thoughts down while they were fresh, including how I thought the lesson went and what changes I would make for a future implementation. While some of the information was very specific to that lesson, some of the pieces were repeated enough to be considered general reflections on the curriculum as a whole.

The first of these is that I saw a great improvement in the areas of getting started on a problem and persistence. When I first began giving my students problem situations as part of this curriculum, many of them lacked the knowledge or confidence to begin on something they had never seen before. Similarly, many of them did not have strategies to continue working if they had trouble or made a mistake, and thus stopped prematurely. As the lessons went on, I noticed that these responses changed dramatically. Students did not stop working simply because they found a challenge. Although it may run contrary to the self-reported student data I collected, this made me confident that I did make a positive change with these lessons. I suspect that this is mostly due to an increased belief by the students that these questions are within their ability to

complete. I also hope that this is evidence of a switch from a Fixed to a Growth Mindset, but I do not have the evidence to support such a statement.

Another recurring theme to my reflections deals with the use of manipulatives. Over the course of the lessons, I introduced several different kinds of physical objects that students could use to help themselves understand the problem and to uncover patterns by making it more tactile and visual. I noticed, however, that many students did not use the manipulatives as I had intended. For example, many of them seemed to think that using the manipulatives helped them to get the answer, and then stopped. In the case of the Tiling Pools Problem (see Appendix F), many students constructed a pool, possibly counted the number of tiles it took, and then moved on without recording the information or looking at the mathematical structure revealed in the process. This kind of mindset robs them of the opportunity to use colored tiles to set things up in different ways, and recognize the patterns that go into the construction of pools of different sizes (see figure 4-4).

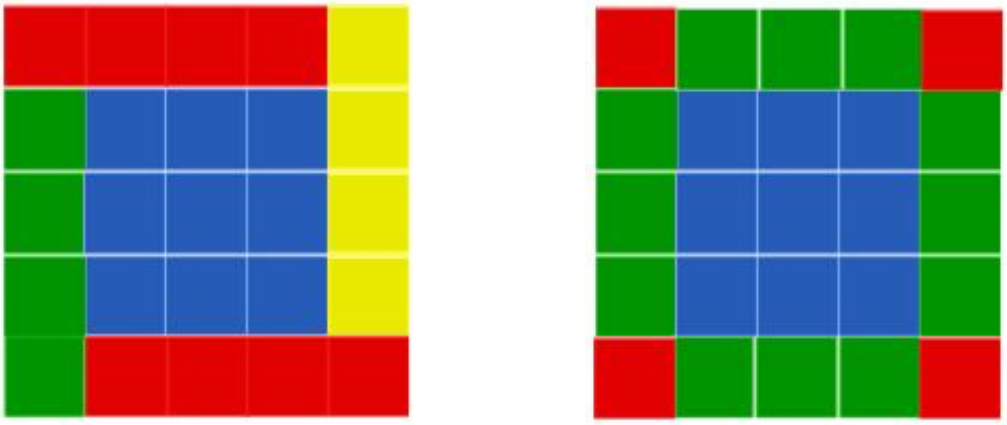

Figure 4-4. Both diagrams show the borders of a pool of side length 3. The left shows how the border tiles are four groups of 3 + 1 tiles. The right diagram shows four groups of 3, with four additional tiles for the corners. Both are valid patterns to recognize in the Tiling Pools Problem.

Another problem with the manipulatives, unfortunately, is that a number of students every time devolved into simply playing with them in a way that did not relate to the problem situation at all. Common themes to this were making towers or stacking them in different ways.

In this way, the manipulatives, instead of helping students understand the problem, provided a distraction and a means to waste time. One possible strategy for helping with this problem is, at least the first time students use a particular manipulative, to give them a couple of minutes to familiarize themselves with them. This gives them time to play with the manipulatives while also allowing them to do things such as categorize them and find possible uses for them.

This leads to the third common theme, that of timing. I consistently wished that I had more time available to me in my lessons. Since I had planned many of the lessons to be parts of a class period (which is usually 57 minutes), I have since revised many of the lesson plans to encompass more of the, if not the entire, period. This increased time allotment did not cause problems this year in terms of covering the required material. In fact, I have covered more material than I have in past years. I attribute this to me being absent fewer days as well as greater efficiency in teaching that comes with experience. It does, however, raise concerns if these lessons were to be taught in another context or by another teacher.

I also realized frequently throughout the curriculum that I needed to put more structure into the student journals that I had them complete following each lesson. The journals themselves were actually a response to my initial journal entry about a need for students to reflect and keep track of the lessons as we progressed through the curriculum. Unfortunately, that meant that I had not fully thought through the use of the journals. I reflected multiple times that I wanted students to have more complete answers to the journals. I also realized that it would help if we kept a running list of problem-solving strategies that we had discussed. This last part is particularly relevant, because the first question I always had them answer as part of their journals was, “What problem-solving strategies did you use while working on this problem?” I felt that the responses this year were overly brief and uninformative. By having a list to choose from,

students would be better able to elaborate on what they did by going through the list and asking themselves if they used that strategy.

Although I had many students who did not put in the full amount of effort or misused the manipulatives, I had some who did a fantastic job, and often approached the problems in a way that I myself would not have considered. There were also expressions of great creativity that I tried to point out and encourage, both to the student in question and to the rest of the class. Some images reflecting these moments of understanding and creativity can be found in Appendix E.

Changes Made

In response to all of this information, I made several changes to my curriculum plan for future implementation. The first change relates to how I collect information about the progress my students make. It is one I had hoped to avoid for purposes of the teacher's time on this curriculum, but it seems crucial, particularly after seeing the results of the questionnaire. I deemed it necessary to create and use a rubric to assess student performance on the skills the curriculum aims to teach them. The rubric grades four categories of student performance: getting started, persistence, use of strategies, and reasonableness of answers. All four of these are addressed in the curriculum at various points, and the student who learns from the lessons taught should grow according to this scale. The category of persistence in particular relates back to the work of Rowlett (2011). A copy of the rubric can be found in Appendix C.

Measuring these four, unfortunately, is more difficult than a simple questionnaire. Judging a student's persistence requires observation while the student is working as well as understanding the process students tend to go through, such as common roadblocks and processes. The categories of getting started and use of strategies can mostly be judged based on the work shown by a student. This can be a problem, however, if manipulatives are used without recording evidence of their use. The reasonableness of answer could be the easiest category to

judge based on work written on a page, but it not only requires a judgment call on what counts as reasonable, it also may require an understanding of the assumptions the student made. This combination of information needing to be gathered while students are working, both from observations and from talking to the students, along with information that can be gleamed from looking at student work, adds up to a large time investment on the part of the teacher. The need for recording observations of particular students during the lesson also takes away from potential guidance and assistance that can be given to students during that time. It was for these reasons that I originally tried to avoid using a rubric for this curriculum.

In addition to the introduction of a rubric as a source of quantitative data, I also deemed in necessary to change the questionnaire. I still think that the survey can provide relevant information, though it needed to be adjusted. I changed the wording of some of the first questions to better reflect my intent, particularly that these strategies are not just used in math class. I also changed the last two problems to be more specifically about problem-solving strategies to get more specific answers that draw on the information they learned throughout the curriculum. I will also preface the questionnaire with an example of a problem in the future. I think that some of the confusion stems from students not knowing, either at the beginning or the end, what is meant by "a problem" in a problem-solving context.

Based on my reflections, I also decided to change several things about the lessons themselves. First and foremost is a decision to devote more time to the lessons, particularly to the discussions and reflections at the end of each one. I found that we ran out of time during the first implementation, and while the activities the students do in order to discover things are very important, it may be more important to devote time to discussion of student attempts and reflection for the students. These steps not only allow for students to cement and record what

they learned, but they give an opportunity for students who struggled with the problem to learn from those who had more success, and to make that success their own. I have revised the timelines for the majority of the lessons to allow for more time, both for those activities and the lessons overall. I also included a running list of problem-solving strategies discussed for student reference with the journals.

I also found that students had difficulty understanding the directions of several of the problems, which has led me to revise them. This encompasses both the directions printed on the page and the directions I give out loud. My hope is that by clarifying the directions without giving away answers, I can lead students to be more likely to work through the problems on their own. Additionally, students pointed out typos and other minor problems to be fixed on the worksheets.

Conclusion

All of this work has led to a revised version of my curriculum (Appendix D). I believe it to be an improvement over the initial version that I implemented in many ways, but I know it to still be a work in progress. I also know that the revisions I made further tailored these lessons to my particular group of sophomores. Anyone who tries to teach this curriculum should keep in mind these important things. Such a teacher will have to adjust things to fit their classes and to make the lessons appropriate for their students, just as I would have to do if I were to teach it to a different group of students. I would also encourage anyone who teaches this to find opportunities and explicitly encourage students to use these skills in other areas of the class. These strategies are meant to be applicable to other situations; students should be shown that.

Before I teach this curriculum again, there is more work on it that I would like to do. I would like to continue searching for problem situations that I can adapt for these lessons. As

time goes on, it may become more likely that my students will have already seen one or more of the problems I have included. Many of the problems were done with most or the entire math department at my school, so it is entirely possible that other teachers will give the same problems in other contexts. As such, finding alternate problems, similar to what I did with Lesson 2 where I have two problems that I can give to students based on their needs, would be helpful. Similarly, I would like to expand the problems that I already have to include deeper levels that students can discover. Not all students will get to these additions, but having more helps to ensure that all students can find the appropriate level of challenge with the problems.

I would also like to put together examples of student work from just about every lesson. Although I may not always find the time to show students examples of what other students have done, it can be a rewarding learning experience. Showing them work samples, real or contrived, can accomplish several things. First, students can learn from the ways others approached the problem, particularly if they differ from the way that student did. Along those same lines, students can learn valuable skills from trying to understand the reasoning of others, whether it is correct or not. This is a prime example of SMP 3: Understanding and Critiquing the Reasoning of Others. One more way that this can be helpful is that it gives students a way to talk about work that is similar to their own. This not only gives them a feeling that others may have tried the same things they did, but it allows them be able to comment on their work without feeling shy about sharing what they did, which is a problem for some students.

The basic tenets of this curriculum are that students can improve their abilities to persevere in, understand, and unravel problem-solving situations. The main methods of making these improvements are to teach strategies for approaching and recognizing patterns, and to expose students to novel situations. By doing this, students can learn critical and creative

thinking skills as well as develop a deeper appreciation for the ways mathematics apply to a myriad of scenarios in many different contexts. These skills are applicable to many settings, and it would be a shame to teach these lessons and rarely give students the opportunities to use what they have learned.

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Appendix A: Unit Plan

Critical and Creative Thinking Skills Curriculum
Length of Unit: 11 weeks (one session per week)
Planning Member: Daniel Albert

1. **Knowledge**: Students will gain experience with:
   - Recognizing and testing assumptions
   - Creating and following a logical plan
   - Testing possible answers to see if they make sense
   - Finding and continuing patterns
   - Recognizing important mathematical patterns
   - Diagramming problems to understand and make clear solution paths

2. **Essential Questions**:
   - How does what we learn in math class relate to problem solving?
   - Where do patterns emerge and how can we use that to our advantage?

3. **Assessments**:
   - A questionnaire will be given at the beginning and end of the 'curriculum' to gauge student thoughts and self-perceptions.
   - Students will be observed as they work towards completing problems, some unlike any they have seen before.
   - A rubric will be used to gauge student abilities and to show growth.

4. **Learning Activities**: Each will be followed by a debriefing focusing on the thinking skills used and the mathematical ideas in the problem or solution paths, including journal questions to be answered individually. In all cases, the first and third questions will be: (1) What problem-solving strategies did you use for this problem? and (3) What would you like to remember about this problem moving forward?
   - Crossing Over the Bridge
   - Connect the Dots/The Doctor's Son
   - Locker Problem
   - Hexagon Dragons
   - Snail Problem
   - Tiling Pools
   - Popsicle Stick Staircase
   - Counting Trains
   - Sneaking up the Lunch Line
   - Golden Apples
   - Crossing the River

5. **Teaching Methodology**: For all of these lessons, it is important that the goal is to have students discover the patterns on their own and think for themselves. This is not to say that no hints, answers, or advice should be given, but rather that the onus must be on the students. When students ask questions, the most common response should be to pose another back at them that helps to guide them to the answer to their question. Additionally, care should be taken to avoid questions or responses that telegraph the answer. For instance, asking for explanations of why a student arrived at the conclusion they did is often better than responding with a right/wrong or asking "are you sure?" An even better situation is to follow up the student's explanation by asking another student (or the class) if they could understand that student's reasoning and if they agree. This not only requires that the student understand their thought process, but also that they be able to communicate it and that other students consider and critique the reasoning of their peers.

Another key thing to emphasize is that there are many possible routes to the correct answer for most of these problems. Various methods of using manipulatives to assist could be equally helpful and many other representations are perfectly acceptable as well. In addition, there are different reasoning paths that will get students to the same conclusion that may all be correct. These are designed so that students can use any of many different methods that suit their style and abilities. That being said, the goal is often not to get the answer to a single question. The manipulatives can be used to simply get an answer, but the intended result of using them is to recognize patterns that would otherwise have gone unseen. Students should be reminded of this and encouraged to try things a different way if all they seem to think about is finding a number for a situation rather than a pattern connecting many situations.

It is also worth noting that these lessons are guidelines that may have worked in a particular setting with particular students. Any teacher who uses these should keep in mind that they can and should be adjusted depending on the setting in which they will be taught. These changes could range from shortening times based on a bell schedule or making more or fewer questions based on the needs of the students for structure. It is highly encouraged that any teacher using this curriculum writes a reflection following each lesson so as to make changes and improve upon it for future implementations.

There are other changes to the classroom and general instruction that are recommended to complement the curriculum. First, it is critical that the class be a safe learning environment where students feel free to make mistakes and to learn without certainty of success. Second, the skills taught in these lessons should be used beyond this curriculum. For example, students could be asked what assumptions they have made about a problem they struggle with after the second lesson on recognizing and challenging assumptions. Another possible change involves have puzzles of various sorts available for students with extra time. These could be used before/after class or after a student finishes an assignment. The availability of puzzles allows students to develop thinking skills in an enjoyable way beyond the lessons and activities of the class.

Appendix B: Student Questionnaire

Please check the box that best represents your agreement with each statement:

	Disagree	Somewhat Disagree	Somewhat Agree	Agree
1. I like math class.				
2. I enjoy learning new things in math.				
3. I always know how to approach new problems.				
4. I see math as useful to my future.				
5. I have learned things in my past math classes.				
6. When I start a problem I've never seen before, I try at least one thing to solve it.				
7. I know several problem-solving strategies.				
8. I have been taught problem-solving skills (ways to solve more than one kind of problem) in school.				
9. I want to do well in school.				
10. If I don't solve a problem on my first attempt, I will keep trying until I can get it.				

Complete each sentence.

11. I feel the way I do about math classes

because_____

_____

_____

12. Some strategies I know to help me begin a new problem include _____

_____

_____

13. Some strategies I know to help me figure out a problem include _____

_____

_____

Appendix C: Problem-Solving Rubric

CRCRTH Curriculum Evaluation Rubric

	4	3	2	1
Getting Started	Student was able to understand the situation and begin working towards a solution appropriately on his or her own.	Student was able to understand the situation and begin working towards a solution by using appropriate tool and/or asking appropriate and specific questions.	Student was able to understand the situation and begin working after several unspecific questions and/or some prompting.	Student was only able to get started after much external prompting.
Persistence	Student worked consistently for the whole time or until an appropriate solution was found, then checked their answer and tried to continue.	Student remained focused for most or all of the time or until a solution was found.	Student showed some persistence, but gave up before finding an appropriate solution.	Student tried something, but stopped well before finding an appropriate solution.
Use of Strategies	Student used multiple strategies, all of which were appropriate to the situation.	Student used one or more strategy, some of which were appropriate to the situation.	Student used multiple strategies, which were not appropriate to the situation.	Student used only one strategy, which was not appropriate to the situation.
Reasonableness of Answers	Student's answer was correct on all parts.	Student's answer was reasonable on all parts.	Student's answer was reasonable in some places, but not others.	Student's answer was not reasonable.

Total Possible Score: _____/16

Appendix D: Lesson Plans

Lesson 1: Understanding a Problem
Problem: Crossing Over the Bridge
SMPs: 1, 3

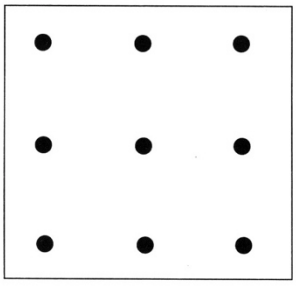
Plan: (50-60 minutes)

1. Teacher gives students the description and gives them time to work on their own. The teacher uses this time to observe and make notes for an initial assessment using the rubric [5-10 min]
2. Teacher reads situation to students twice, then asks for an analysis on the problem (aka how does the situation shape the problem) [5 min]
3. Class acts out one example [5 min]
4. Students are given the worksheet and time to work [10-15 min]
5. Class reconvenes and discusses results. What was the lowest time anyone found? How did people approach the problem? [10 min]
6. Class lists which thinking skills we used/what we learned. Did acting it out help us understand? What other things could we have tried? [10 min]
7. Journal—What did you learn that helps you understand a problem? [5 min]

Lesson 2: Recognizing and Challenging Assumptions
Problems: Connecting the Dots, The Bunny Box, and Doctor's Son
SMPs: 3

Plan: (45 min)

1. Students are given whiteboards and dry erase markers. The shape below is projected onto the board and students are given the following directions: Draw straight lines to connect all of the nine dots without lifting your marker. You may change direction only three times. Students who correctly get the answer (or if there is extra time) are given the rabbit question as well.[5-10 min]



The Bunny Box

Nine rabbits share a square enclosure at the pet store.

Build two more square enclosures to give each rabbit a pen of its own.

2. After several minutes, students are asked how they did. A couple students who did NOT get the answer describe to the class how they approached the problem. [5 min]

3. Students are then given the following situation: [3 min]
   A father and his son are in a car accident. The father dies at the scene and the son is rushed to the hospital. At the hospital the surgeon looks at the boy and says "I can't operate on this boy, he is my son" .... How can this be??
4. Students are asked how quickly they discovered an answer (there are multiple). Those that had trouble are asked what assumptions they made going in. Students who got an answer then describe their thought process. Knowing one solution, students are asked for additional ones. [7 min]
5. The teacher gives a definition of assumptions and a discussion of assumptions occurs in which students are asked to generate assumptions they have or make on a daily basis. Discuss whether or not it's okay to make assumptions and how to recognize them when we do. Students are given a few more minutes to work on the problems if they need.[15 min]
6. Journal—What did you learn about the role assumptions play in problem solving? [5 min]

Lesson 3: Representing Problem Situations

Problem: Locker Problem
Mathematical Concept Introduced: Perfect Squares
SMPs: 1, 5, 7, 8

Plan: [40 min]

1. Students are given the Locker Problem and are told to get started. [3 min]
2. Teacher checks in and asks how students are approaching the problem. If no one mentions having trouble keeping track of things, the teacher prompts about this specifically. Manipulatives are made available for students to continue. The teacher circulates, asks guiding questions, particularly of students who have run into trouble or made a mistake. [7-12 min]
3. Students are asked what they found. Did they find a pattern? Can anyone explain the pattern? Students are introduced to the idea of 'perfect squares' and how factors relate to them. [5 min]
4. Students are asked how they represented the problem. Several students are given the opportunity to share. Possibilities shown should include both with and without manipulatives. Students are made aware that manipulatives will be available from now on. [5 min]
5. Problem-solving strategy introduced: Start with smaller numbers. The problem began with 20, which is reasonable to work out, then does 30 which is also reasonable, but then jumps to 200, which is not. [4min]
6. Problem-solving strategy introduced: Use manipulatives. The point of the manipulatives is NOT to get the answer. The point is to make the problem more visual and tactile to help you see patterns you might otherwise miss. [4min]
7. Journal—What is the point of starting with smaller numbers and using manipulatives? [5 min]

Lesson 4: Modeling Situations
Problem: Hexagon Dragons
SMPs: 1, 2, 4, 5, 8

Plan: [35 min]

1. Students are given the Hexagon Dragon worksheet to work on. [5 min]
2. Teacher checks in with the class about patterns found and what the future Dragons would look like. [3 min]
3. Students given more time to work. [14 min]
4. Teacher leads a discussion on modeling situations with equations. Students are given the chance to explain how they approached the problem and what their equations (or expressions) were. There are many valid ways to do this, and the similarities should be pointed out. It should also be made clear that all of the equivalent methods are still correct. [8 min]
5. Journal—What did you learn about representing a situation with an expression or equation? [5 min]

Lesson 5: Frames of Reference
Problem: Snail Problem
SMPs: 1, 2, 4, 5, 6, 8

Plan: [45 min]

1. Give students the Snail Problem to work on. Teacher circulates and checks answers (many will be incorrect) and prompts students to represent the problem in a different way. [10 min]
2. Teacher stops work and first asks students how they represented the problem and what their process was. Students are NOT asked for the answer at first. Several different methods should be gathered. [7 min]
3. Students are then asked what differences exist between the methods. Are any better than others? Why? [5 min]
4. Students are given more time to work. [10 min]
5. Now students are asked for the answer. Students are prompted to make the connection between the different methods and the troubles in finding the correct answer. Teacher explains that the way solutions are framed sets up a frame of reference that can help or hinder the obtaining of the correct solution. None of the methods were wrong, but how we interpreted them might have been. [8 min]
6. Journal-- What did you learn about how perspective affects a problem as you worked on this? [5 min]

Lesson 6: Review of Representing Problems, Modeling, and Frames of Reference
Problem: Tiling Pools
Mathematical Connections: Equivalency of Expressions
SMPs: 1, 2, 4, 5, 6, 7

Plan: [40 min]

1. Students are given the Tiling Pools problem and instructed to work on it either individually or in pairs. If necessary, they will be reminded that manipulatives are available. Observations are made for mid-curriculum rubric assessment.[15-20 min]
2. Students are grouped into fours to discuss their answers. [5 min]
3. The class then discusses answers and is given the opportunity to share how they came up with the equation. If possible, discussion focuses on the different ways that the equation could be represented and how they are all equivalent. For example, $4x + 4 = 4(x + 1) = 2x + 2(x + 2) = (x + 2)^2 - x^2$. [10 min]
4. Journal—Which of the ways of representing the situation made the most sense to you? Could you still understand the others? [5 min]

Lesson 7: Recognizing Patterns
Problems: Popsicle Stick Staircase
Mathematical Concept Introduced: Triangular Numbers
SMPs: 1, 2, 4, 5, 7, 8

Plan: [50 min]

1. Students are given the Staircase Problem and instructed to work either individually or in pairs. If necessary, they will be reminded that manipulatives are available. [15 min]
2. Class discussion about questions up to and including #5. Discussion includes strategies or methods used. [8 min]
3. Students are given more time to work on numbers 6-9. [10 min]
4. Class discussion about progress on these new questions. Building on any student success, triangular numbers are introduced and the formula given. [12 min]
5. Journal—What did you learn about triangular numbers while working on this problem? [5 min]

Lesson 8: Testing your answer
Problem: Sneaking up the lunch line
Mathematical Concept Introduced: Floor Function
SMPs: 1, 2, 3, 4, 6

Plan: [35-50 min]

1. Teacher will read the Sneaking up the Lunch Line problem twice, then asked to summarize. The class acts out two different instances. [5-10 min]
2. Students are given the problem worksheet to work on individually or in pairs. Teacher gives hints and advice along the way. Most commonly, prompting students to check their hypotheses to ensure that they fit the situation. [10-15 min]

3. Students are asked what progress they made. Were there any patterns they thought they saw which they later disproved? How did they go about testing their ideas? [10 min]
4. Ideally, after a student suggests the pattern of 'divide by three, round down', the concept of a floor (or ceiling) function is introduced. [5-10 min]
5. Journal—What did you learn about testing your ideas? [5 min]

Lesson 9: Working Backwards; Guess, Check, Generalize
Problem: Golden Apples
Mathematical Connections: Equations of Lines
SMPs: 1, 2, 4, 7, 8

Plan: [45 min]
1. Students are given the Golden Apples Problem and given time to begin. Teacher circulates and asks questions and gives hints to students. [20 min]
2. Class discussion about the methods for finding the number of apples with which the Prince started. Most will have used Working Backwards or Guess and Check. [5 min]
3. Discussion about the ways to create an equation. One method is to create an equation given two points. [5 min]
4. Introduce the idea of Guess-Check-Generalize (and how it's different from and better than Guess and Check for this situation). [10 min]
5. Journal—What did you learn about generalizing/making a shortcut? [5 min]

Lesson 10: Additional Experience and Connections
Problem: Counting Trains
Mathematical Concept Introduced: Exponential Growth, Pascal's Triangle
SMPs: 1, 2, 3, 4, 5, 6, 7, 8

Plan: [40-50 min]
1. Split students into pairs. Give students Cuisenaire Rods and allow them to explore or "familiarize" themselves with the rods. Discuss what students noticed, particularly color coding as it relates to length. [3-5 min]
2. Introduce the Train Problem and how we are paying attention to the length of the train, the number of combinations, and the number of ways you can build a train with the same components. Ask how they can use the Cuisenaire Rods and discuss their use as representatives of train length. Build all combinations of train lengths 1, 2, and 3. Discuss whether or not order matters. Model a method of recording the combinations. [8-10 min]
3. Set the task of building all trains of length 4 and 5. Require that one student in each pair record the combinations the other builds, but remind them that they should work together. Teacher circulates and ensures students are building and recording. Prompt students to consider whether or not they have all of the combinations and how they could be sure. Suggest putting the combinations in an order to help.[10 min]
4. Collect all of the combinations of length 4 and discuss ordering them. Collect responses for the number of combinations of length 5. [5 min]
5. Make a table and discuss a pattern for the number of combinations of trains of a given length. Introduce and elaborate upon exponential functions [10-15 min]

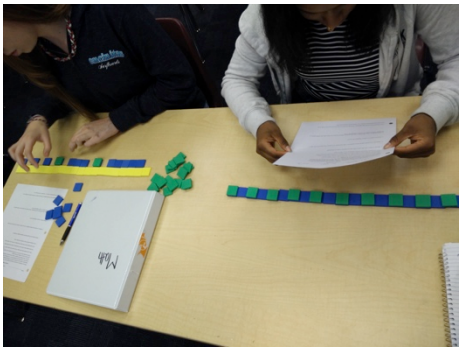
6. Journal—What patterns did you notice while working on this problem? [5 min]

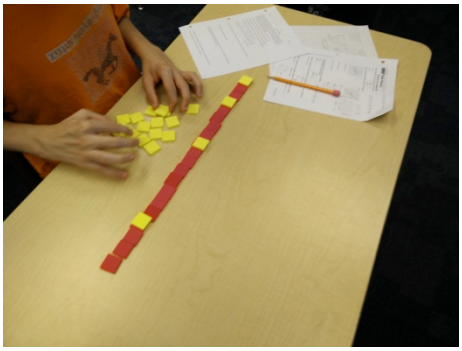
Lesson 11: Seeing How Far We've Come
Problem: Crossing the River
SMPs: 1, 3, 4, 5, 6, 8

Plan: [40 min]
1. Students are given the worksheet and allowed to progress. The teacher observes for a final assessment by the rubric while giving prompts to students who require them. [25 min]
2. Class discussion about methods, patterns found, and ways they discovered a rule. [5 min]
3. Journal—Think back to the first problem we did—Crossing over the Bridge. How have you changed or improved over the course of this curriculum? [10 min]

Appendix E Selected Images

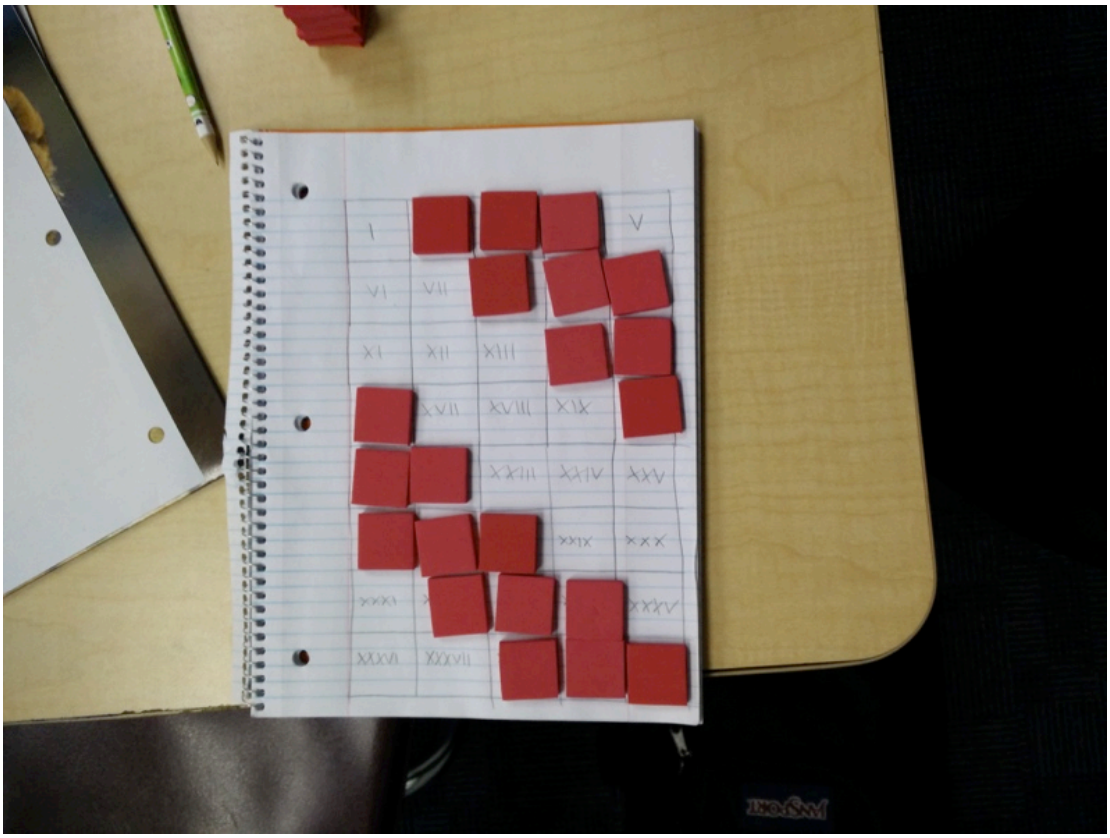
Lesson 3: The Locker Problem



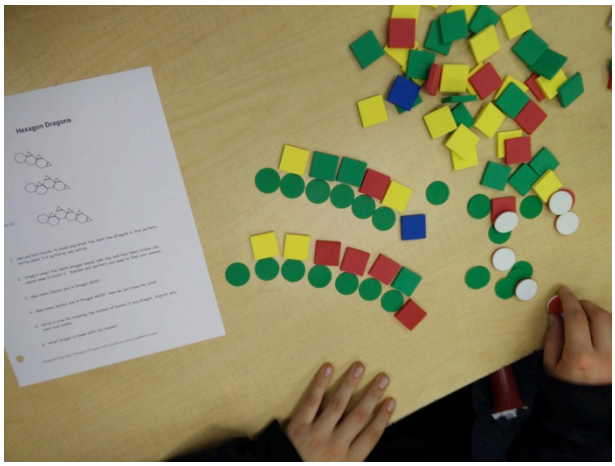


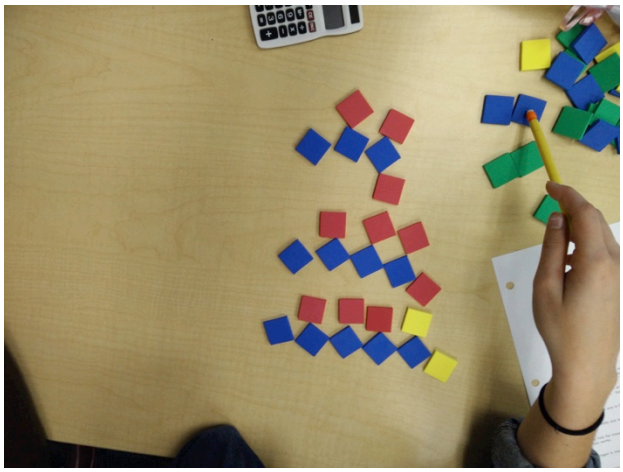
The above two images show students using similar methods. All three are using the colored blocks to represent lockers and the different colors to show whether those lockers are opened or closed. The students change the color of the appropriate locker as they go through the process outlined in the problem.

The student in the image below took another approach. She had drawn a square to represent each of the lockers and numbered them. She then either covered or uncovered the number based on whether the locker was open or closed. She had extended the problem beyond the original 20 lockers at my direction in order to see if the pattern she discovered held.



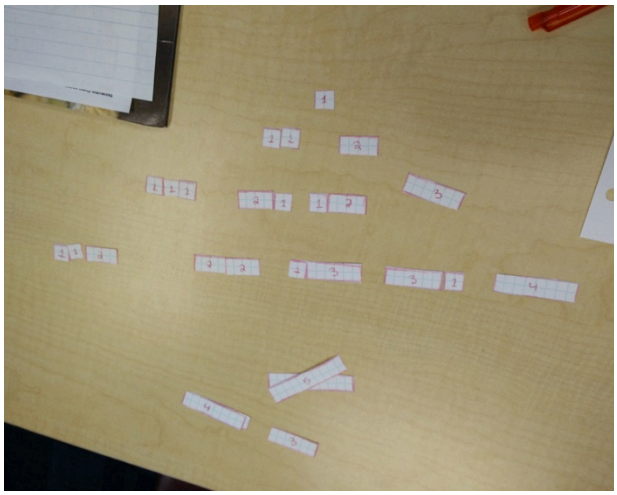
Lesson 4: Hexagon Dragons





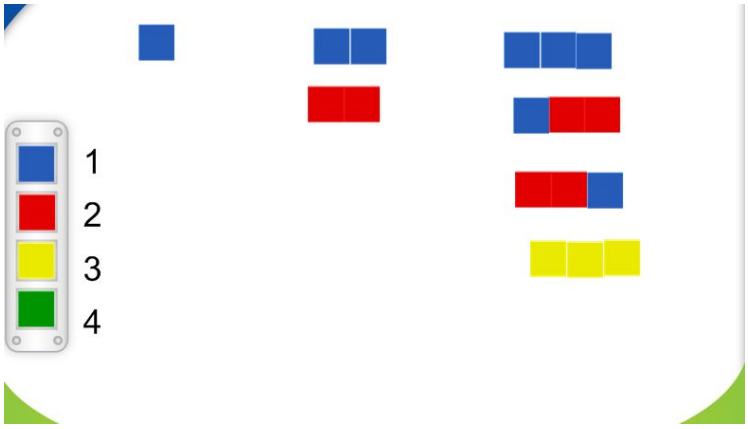
Both of the above students took similar approaches to the problem. They decided to build the dragons out of blocks and/or disks. They used either the different shapes or different colors to represent the hexagon and triangle segments in the problem. They were searching for patterns, both in the number of pieces used and in the ways in they could create the next diagram from the previous.

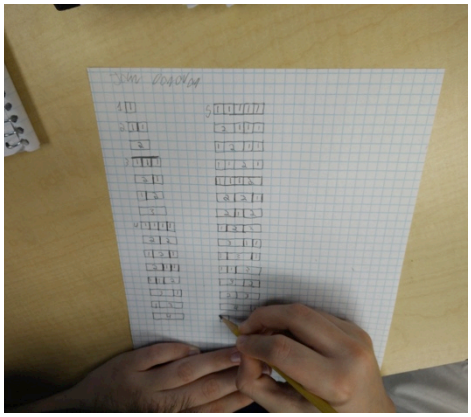
Lesson 10: Counting Trains



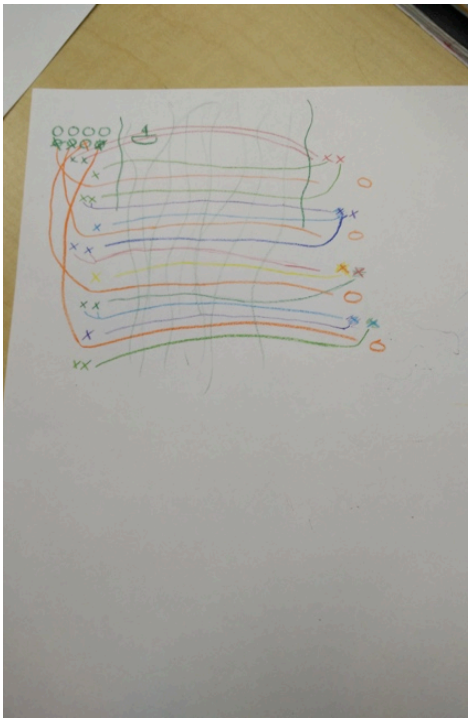
Both of the above students decided to build the various combinations of trains with physical objects. The top left student used the blocks they had become familiar with throughout these lessons while the top right student used pieces of paper of different lengths. They both were looking to count the number of ways to build the trains, though I detect a partial pattern to the combinations in the top right picture.

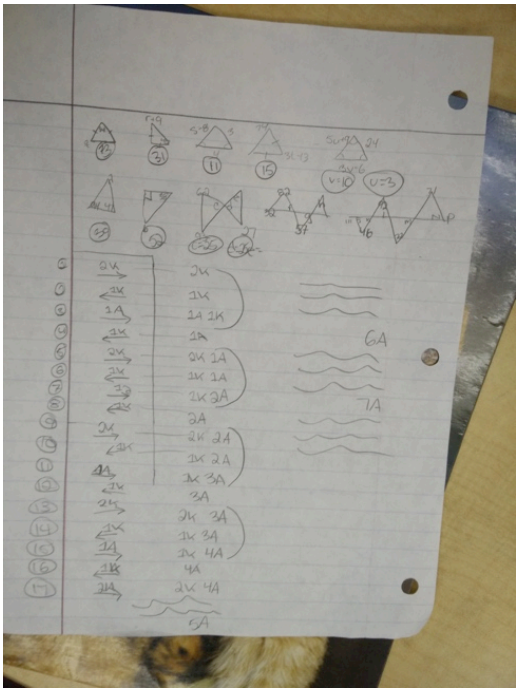
The bottom left picture actually shows the list of combinations that I created. I projected this on the board during our discussion before students got to work on the problem. My purposes in doing so were twofold. It not only ensured that students understood what the problem meant, but also demonstrated one way to show the different combinations using the blocks. The blue blocks represent trains of length 1, the red blocks represent trains of length 2, and so on. The lower right picture shows a student who decided to list out the combinations using graph paper. I encouraged students (in all of the lessons) to pick a method they felt most comfortable with and that resonated with their thought processes.





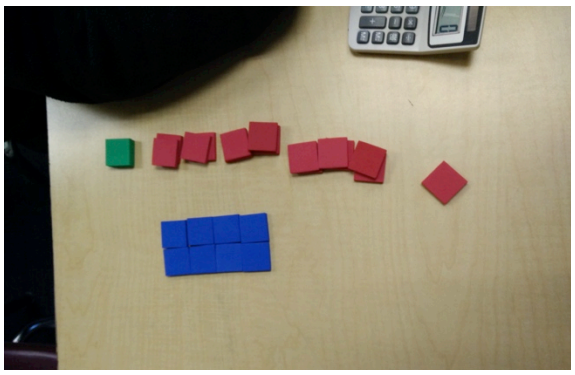
Lesson 11: Crossing the River





Each of the above students tried to diagram the situation using paper. Though their methods varied, the purpose was the same. They wanted to discover how many trips it took to get everyone across.

The student below tried to use the blocks to represent this. The green and red blocks represent children and adults, respectively. The student moved them from one side of the blue blocks (the river) to the other while counting how many trips it took. All three of these examples showed students finding repetitive processes that led them to finding patterns and shortcuts, a clear example of SMP #8.



Appendix F Problem Worksheets

Lesson 1: Crossing Over the Bridge

Crossing Over

The Situation:

Four people come to a river in the night. There is a long, rickety bridge with missing planks that can only hold two people at a time. Only one of them has a phone whose battery isn't dead. Because it's night and the footing on the bridge is uncertain, they need the phone to be used as a flashlight when crossing the bridge. Agatha can cross the bridge in one minute, Beatrice in two minutes, Clyde in five minutes, and Dwight in eight minutes. When two people cross the bridge together, they must move at the slower person's pace.

The Question:

What is the least amount of time it could take them all to cross the bridge? How is this done?

Lesson 3: Locker Problem

Name _____ Date _____ Period _____

Locker Problem

There are 20 lockers in the hall of Kennedy Middle School. In preparation for the beginning of school, the custodian cleans the lockers and paints fresh numbers on the locker doors. The lockers are numbered from 1 to 20. When the 20 students from Mr. Newton's class return from summer vacation, they decide to celebrate the beginning of the school year by working off some energy.

• The first student runs down the row of lockers and opens every locker.

• The second student starts with locker #2 and closes every second locker.

• The third student starts with locker #3 and opens or closes every third locker.

• The fourth student starts with locker #4 and opens or closes every fourth locker.

• This continues until all twenty students have taken a turn.

1. Which lockers are still open after the twentieth student is finished?

2. What do you notice about these numbers? Can you think of a reason why these are still open?

3. Which locker or lockers changed the most?

4. Suppose there were 30 lockers and 30 students. Which lockers are open after the 30th student is finished? Which locker or lockers changed the most? What about 200 lockers with 200 students?

Task adapted from Fostering Algebraic Thinking A Guide for Teachers Grades 6-10 page 10 by Education Development Center, Inc. Heinemann, Portsmouth, NH

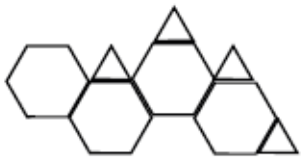
Used with permission. Credit goes to Foulser Education Consulting.

Lesson 4: Hexagon Dragons

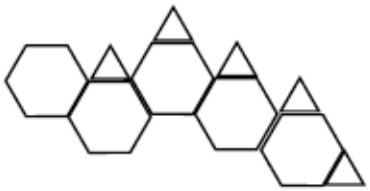
Hexagon Dragons

Dragon #1

Dragon #2



Dragon #3



1. Use pattern blocks to build and draw the next two dragons in this pattern. Write down 3-5 patterns you notice.

2. Predict what the tenth dragon would look like and how many blocks you would need to build it. Explain any pattern you used to find your answer.

3. How many blocks are in Dragon #25?

4. How many blocks are in Dragon #100? How do you know for sure?

5. Write a rule for knowing the number of blocks in any dragon. Explain why your rule works.

6. What dragon is made with 126 blocks?

Adapted from the *Hexagon Dragon* task found on www.mathwire.com

Used with permission. Credit goes to Foulser Education Consulting.

Lesson 5: Snail Problem

Name:_____ Date:_____ Period:____

Snail Problem

A snail is trying to climb up the wall of a building. The wall is 39 feet high. Every day, the snail climbs 9 feet. Every night as it sleeps, the snail slides down 6 feet. The snail starts at the bottom of the wall. How many days does it take the snail to reach the top of the wall?

Show your work/describe your method here:

The most common first answer 13. Why do you think that is? Why is this not correct?

Make an equation for where the snail begins on day x.

Make an equation for how high the snail gets on day x.

What is the difference between these equations? Is this difference important to the problem?

How many days would it take the snail to climb to the top of an 83-foot tall building?

How many days would it take to climb to the top of a 45-foot tall building if the snail climbed 11 feet each day and slid down 7 feet each night?

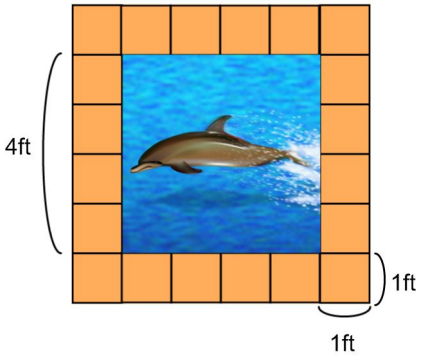
Can you make a general equation for an h-foot tall building if the snail climbs c-feet every day and slides down s-feet every night?

Lesson 6: Tiling Pools

Name _____ Date _____ Period _____

Tiling Pools Problem

Hot tubs and in-ground swimming pools are sometimes surrounded by borders made out of tiles. This drawing shows a square pool with sides of length 4 feet surrounded by square border tiles. The border tiles measure 1 foot on each side. A total of 20 tiles are needed for the border.



This is the pool of side length 4. How many tiles are there?

1. Sketch or otherwise create **square** pools of side length 1, 2, 3, 5, 6, and 10. Make note of how many border tiles are required for each.

2. Record three patterns you noticed when creating the pools. Think about how you built them and the relationship between the numbers of tiles.

3. How many tiles would you need for a pool of side length 99? What size pool would have 236 border tiles?

4. Write an equation for the number of tiles based on the side length of the pool.

5. What do the numbers in your equation represent? Is there a visual way to demonstrate your equation?

Task adapted from the "Tiling Pools" problem in Investigation 1 of CMP2 unit *Say it with Symbols* published by Pearson Prentice Hall.

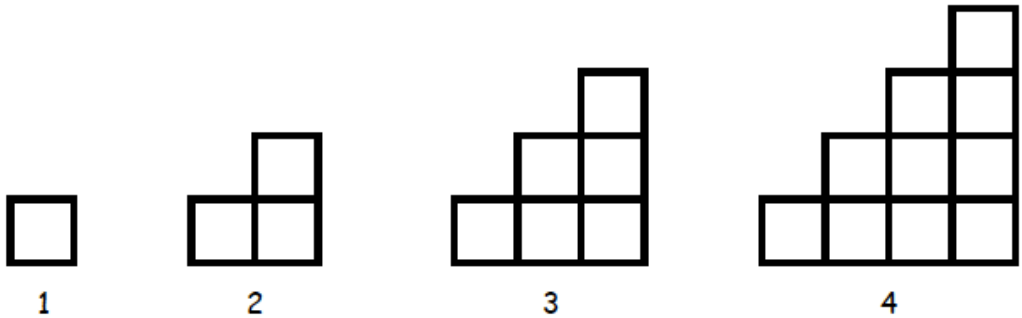
Used with permission. Credit goes to Foulser Education Consulting.

Lesson 7: Popsicle Stick Staircase

Name: _____ Date: _____ Period: ____

Popsicle Stick Staircase Problem

Below are four staircases made up of popsicle sticks.



1. Draw or build the next two staircases in this pattern.

5 6

2. Write down 3-5 patterns you notice.

3. Perimeter sticks are the ones that go around the outside of each staircase. How many are there in staircase 10? Try to do this without building or drawing it.

4. Write a rule for knowing the number of perimeter popsicle sticks for any staircase. How do you know that your rule works?

5. What staircase has 72 perimeter sticks?

6. Look at the number of small squares in each shape. Do you notice any patterns?

7. How many small squares make up staircase 7? What about staircase 10?

8. Write a rule for knowing the number of small squares that make up each staircase. Explain why your rule works.

9. How many blocks are in staircase 100? How do you know for sure?

10. How many <u>*total*</u> sticks do you need to make staircase 10? Staircase 20? Can you write a rule for this as well?

Problem adapted from CMP *Say It With Symbols*, Page 62-63

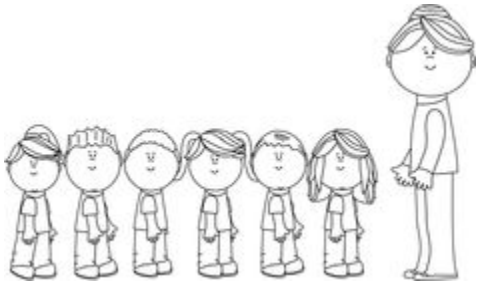
Used with permission. Adapted from "Toothpick Staircase" from Foulser Education Consulting.

Lesson 8: Sneaking Up The Lunch Line

Name: _____ Date: _____ Period: _____

Sneaking Up The Lunch Line

Billy Sneak is at the end of the lunch line, but he's too impatient to wait his turn. Every time someone checks out and buys their lunch, he cuts in front of the two people ahead of him. If there is only one person (or no one) in front of him when he would skip people, he just moves to the front of the line.



1) Billy is at the back of the line with 50 people ahead of him. Estimate (educated guess) how many people will buy their lunch before he gets to the front of the line.

2) Work through and figure out how many people will actually buy their lunch before Billy if there are: (show your work or use manipulatives)
   a) 6 people in front of him.

   b) 11 people in front of him.

   c) 16 people in front of him.

3) Explore some other numbers and record your results here.

Number in front of Billy						
Number who buy before Billy						

4) Do you see a pattern? Describe how you can find the number or people who buy their lunch before Billy without doing it out completely.

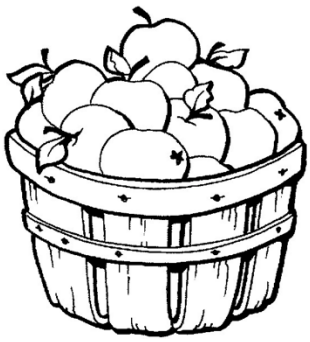
5) Fill in the blank spaces in the table using the pattern you described above:

Number in front of Billy	37	296	1000			7695
Number who buy before Billy				13	26	

6) Billy gets more and more impatient! Explore how the pattern changes if he cuts in front of 3 people at a time, or 4 at a time, or even 10 at a time!

Used with permission. Adapted from "Sneaking up the Line" from Foulser Education Consulting.

Lesson 9: Golden Apples



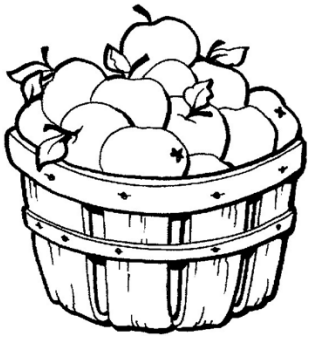
Golden Apples

A prince picked a basketful of golden apples in the Enchanted Orchard. On his way home, the prince was stopped by a troll who guarded the orchard. The troll demanded payment of one-half of the apples plus two more. The prince gave him the apples and set off again. A little further on, he was stopped by a second troll guard. This troll demanded payment of one-half of the apples the prince now had plus two more. The prince paid him, and set off once more. Just before leaving the Enchanted Orchard, a third troll stopped him and demanded one-half of his remaining apples plus two more. The prince paid him and sadly went home. He had only two golden apples left. How many apples had he picked?

Disappointed at having only two apples left, the prince set out the next day to increase his bounty. He encountered the same three trolls, who made the same demands. He ended the day with five apples left. How many apples did the prince pick on Day 2?

Persevering in his quest for more apples, the prince tried for a third day, this time with a goal of returning to his castle with enough apples for the entire royal family. He arrived home with 11 apples. How many apples did he pick on Day 3?

Help make the prince's life easier: If he has a goal of ending the day with **n** apples, how many apples must he pick?



Platinum Apples

To outsmart the trolls, the prince heads to a new orchard and picks a basket of apples. Alas, the new orchard is also enchanted, and the prince encounters three different trolls guarding the bridges to the exit. The first troll demands payment of **one-third** of the prince's apples plus **three** more; the second troll demands one-third of the remaining apples plus **two** more; the third, one- third of the prince's remaining apples plus **one** more.

The prince exits the orchard with seven apples. How many did he pick?

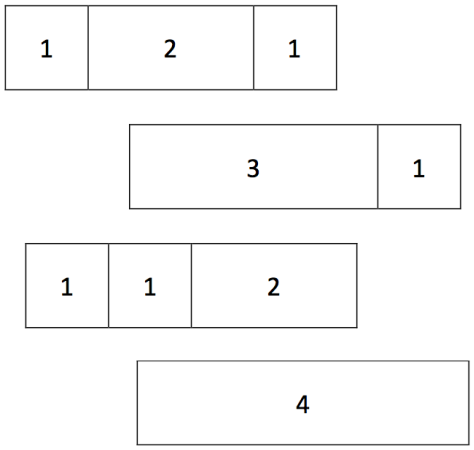
"Golden Apples" was adapted from Driscoll, M. *Fostering Algebraic Thinking*, Heinemann 1999. "Platinum Apples" was written by DMPA Course Facilitators Ellie Goldberg and Kathy Foulser.

Used with permission. Credit goes to Foulser Education Consulting.

Counting Trains

Adapted from the *Trains of Thought* task found in Ways to Think About Mathematics by Benson, et al., published jointly by Corwin Press and Education Development Center.

Cuisenaire rods can be used to build trains of a given length. Below are examples of "trains of length 4".



Notice the 1-2-1 train and the 1-1-2 train are made up of the same rods. However, since the order of the rods are different, they are considered two different trains.

Part I

7. Use the Cuisenaire rods to make all of the trains of length 1, 2, 3, 4, and 5. Make a list of all the trains using any notation that works for you. As you build the trains think about and write down how you know you have made them all and how you know you didn't make the same train more than once.

8. Write a set of directions that someone else could use to make all the trains of a given length.

Part II

3. How many trains are there of

   a. Length 1

   b. Length 2

   c. Length 3

   d. Length 4

   e. Length 5

4. Write a rule for determining the number of trains of any length.

5. How do you know your rule will work for any length train? (Hint: think about how you constructed and listed your trains.)

Part III

Now think about the number of ways that a train of a given total length

Number of cars in the Train ➔ Total Length of Train ➓	1	2	3	4	5	6
1						
2						
3						
4						
5						

can be built with a given number of cars.

1. Fill in the table above.

2. How many trains of length 12 have 3 cars?

3. How could you determine the number of trains of a given length with a given number of cars?

Adapted from "Trains of Thought," found in *Ways to Think About Mathematics: Activities and Investigations for grade 6-12 Teachers* by Steve Benson, published by EDC and Corwin Press.

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Lesson 11: Crossing the River

Crossing the River

A group of 8 adults and 2 children need to cross a river. They have a small boat that can hold either 1 adult OR 1 child OR 2 children.

1. How many one-way trips does it take for all 8 adults and 2 children to cross the river? Show or explain how you got them all across.

2. How many one-way trips would it take to get the following groups across the river

   a. 6 adults and 2 children

   b. 15 adults and 2 children

   c. 3 adults and 2 children

3. How many trips would it take to get 100 adults and 2 children across the river? Describe how you found your answer?

4. Write a rule for finding the number of trips needed to get any number of adults (A) and 2 children across the river.

5. It takes 41 trips to get all the adults and the 2 children across the river. How many adults where in the group?

6. What happens to your rule for finding the number of trips if there are a different number of children? For example: 8 adults and 3 children or 2 adults and 5 children?

7. Write a rule for finding the number of trips needed to get any number of adults (A) and any number of children (C) across the river.

8. One group of adults and children took 27 trips.

   a. How many adults and children where in the group?

   b. Is there more than one solution to this question?

   c. If so, what rule fits each solution?

Task adapted from *The Fostering Algebraic Thinking Toolkit* Session I pages 15-17 by Education Development Center, Inc. Heinemann, Portsmouth, NH

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