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Hydrodynamic Analogues of Hamiltonian Systems

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HYDRODYNAMIC ANALOGUES OF HAMILTONIAN SYSTEMS

A Thesis Presented

by

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ABSTRACT

HYDRODYNAMIC ANALOGUES OF HAMILTONIAN SYSTEMS

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A one-dimensional Hamiltonian system can be modeled and understood as a two-dimensional incompressible fluid in phase space. In this sense, the chaotic behavior of one-dimensional time dependent Hamiltonians corresponds to the mixing of two-dimensional fluids. Amey (2012) studied the characteristic values of one such system and found a scaling law governing them. We explain this scaling law as a diffusion process occurring in an elliptical region with very low eccentricity. We prove that for such a scaling law to occur, it is necessary for a vorticity field to be present. Furthermore, we show that a conformal mapping of an incompressible fluid in an annular region to an elliptical region explains these results for any positive eccentricity less than one.

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CHAPTER 1

INTRODUCTION

The power of Hamiltonian mechanics lies in the fact that the dynamics of a system, however large and complex, can be described by a single function, the Hamiltonian. All the information necessary to understand the future and past of the system are described by a time-independent Hamiltonian and its state at a specific moment, the initial conditions. As the Marquis of Laplace put it:

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes." (Laplace, 1902).

In the Laplacian view of the universe, the only uncertainty that exists is due to our lack of capacity to measure the current state of the universe with exactitude. In fact, we can construct statistical mechanics by considering probability distributions on the initial conditions and by finding average properties of the system, such as pressure and temperature. But, alas, chaos theory and quantum mechanics have shown us that Laplace's view is fundamentally erroneous. Chaos theory shows that deterministic systems could be so sensitive to the initial conditions that it may be impossible to know their state in the future with precision and that, in fact, for all practical purposes, it is stochastic in the long term. The only way to overcome this difficulty is to measure the initial conditions with infinite precision. This is where quantum mechanics shows us that there is no such thing as an infinite precision measurement. Heisenberg's Uncertainty Principle tells us that there is a limit to the precision of any measurement. We cannot simultaneously and with infinite precision determine the position and the speed of a particle. The precision is limited by Planck's constant h, approximately 6.62606957E-34 Js.

A further objection to Laplacian determinism can be found in computer science, which establishes that to represent with infinite precision a transcendental number, we would need an infinite number of bits. Since the amount of bits in the universe is finite due to the fact that as far as we know the universe is finite, Laplace's vast intellect will be in a pickle if at some point in its calculation it finds a transcendental number.

Although Laplace's vision of mechanical determinism seems naïve from the point of view of a 21^{st} century scientist, it is still useful because even in the worst chaotic system there are conserved quantities, known as constants of motion.

The most notable consequence of the Hamiltonian theory is the existence of other functions of the state of the system that are conserved quantities. These quantities can be calculated for the initial conditions and will be the same for every other state of the system in the future and in the past. We call some of these conserved quantities *Laws of Physics*. The first notable conserved quantity is the Hamiltonian itself, which produces the *Law of Conservation of Energy.* A few more are: the *Law of Conservation of Angular Momentum*, the *Law of Conservation of Impulse (Newton's Laws), Law of Conservation of Mass, Law of Conservation of Charge,* etc. The number of conserved quantities is different from system to system. For example, solitons, produced by the Korteweg-de Vries equation, have an infinite number of conserved quantities (Morrison 1998).

The state of a Hamiltonian system composed of a finite number of particles is represented by a vector of size 2 times the number of dimensions of the system times the number of particles. For example, a long rigid pendulum will be described only by two numbers, the angle with respect to the vertical and the angular speed. The state of a planet orbiting the Sun will be described by its Cartesian coordinates on the orbital plane and its velocity vector on the same plane, four numbers. The state of n point vortices in the surface of a sphere is described by the latitude and longitude of each vortex, 2n numbers.

A system is integrable by quadrature (or just integrable) when the size of the state vector is twice the size of the number of constants of motion. Single particle systems with only one spatial dimension like the pendulum are always integrable because the Law of Conservation of Energy is always a constant of motion. The motion of a planet is always integrable because the angular momentum and energy are conserved quantities. The motion of two planets that are close enough to interact strongly is not integrable because the state vector is size eight and the best we can do is to produce three constants of motion, the magnitude of the Lenz-Runge vector being the source of the additional two. The vortex system is integrable up to three vortices because there are three constants of motion, including energy conservation specific to that system.

Non-integrability is a necessary but not sufficient condition for chaotic behavior. Kolmogorov, Arnold and Moser (KAM) proved a theorem that establishes that for small enough perturbations of an integrable system, the resulting trajectories are quasi-periodic, and consequently non-chaotic. An exceptionally clear application of this theorem is the motion of the Solar System composed of the Sun, eight planets and an immense number of minor bodies as large as the Earth's Moon or Ganymede. As far as human history is concerned, the Solar System's behavior has been completely deterministic. So much so that we measure time based on its periodic motion. Nevertheless, it is non-integrable given that its state vector size exceeds, by far, the number of constants of motion. The loophole created by the KAM theory manifests itself as orbital resonance between bodies that are close enough to each other to affect their orbits strongly (Peale, 1976). An orbital resonance means that the ratio of the period of two objects is the ratio of two integer numbers. For example, the moons of Jupiter (Io, Europa and Ganymede) have a ratio 4:2:1. Although long time simulations of the Solar System have shown it to be stable to the currently accepted orbital element numbers and masses, it has been proven that small changes in these parameters produce large long term effects in the orbits. Even in some scenarios, Mercury falls into the Sun. This sensitivity to small changes in the initial state demonstrates that the Solar System is essentially chaotic and that small perturbations (for example, a comet) may produce large changes in its dynamics in the long term. Jaques Laskar calculated the Lyapunov exponent of the Solar System to be 1/(5Myear) (Laskar, 1989). The Lyapunov exponent measures how initially close trajectories separate with time.

Another example of regular motion due to the KAM theorem is the Fermi-Pasta-Ulam (and Tsingou) experiment. Considered the birth of Computational Physics, the FPU experiment in 1954 tried to produce stochastic motion from a deterministic Hamiltonian (Daxois et al, 2005). To the surprise of all involved, it turned out that they produced regular motion. These results were considered paradoxical if not erroneous until 1965, when Zabusky and Kruskal proved that, in the continuum limit, the FPU system dynamics were governed by the Korteweg-de Vries equation, starting the modern era of soliton research.

Even though much of the discipline we call Physics resides in the knowledge of conserved quantities, their emergence didn't become clear until the early 20^{th} century thanks to Noether's theorem. Emmy Noether was a rarity in her time, a professional woman Mathematician. Einstein said of her:

"In the judgment of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began." (Neuenschwander, 2011).

Noether's theorem proves that symmetries of the space that contains a physical system produce conservation laws. For example, the fact that a ball falls at the same speed if I drop it right now or if I drop it at the same place one hour later tells us that the dynamics of the falling ball are independent of the time I choose to drop it. This symmetry of time translation tells us, thanks to the Noether's theorem, that energy is a conserved quantity. The symmetry respect of space translations and rotations means that the system is independent of where I choose the origin of coordinates or how I orient it. These symmetries tell us, through the Noether's theorem, that the impulse and angular momentum are conserved quantities. In quantum mechanics, system independence from gauge transformations (changes on the phase of the wave function) tells us among other things that electric charge is a conserved quantity.

A one-dimensional Hamiltonian system is isomorphic to an incompressible fluid in two dimensions. Louisville's theorem tells us that the phase space volume of a Hamiltonian system is conserved just like in an incompressible fluid. In order to understand them, fluids are approached from two equivalent points of view, Eulerian and Lagrangian.

The Eulerian approach associates a velocity vector with every point of the space containing the fluid. We can think of the Eulerian approach as a man on a bridge observing a river passing beneath him. This man will be concerned only with the velocity of the flow at particular points, for example at the base of the bridge.

The Lagrangian approach follows the trajectory of individual particles in the fluid. In this case the man on the bridge will focus on a particle in the fluid, a leaf for example, and follow its trajectory. Both Lagrangian and Eulerian approaches are mathematically equivalent and the only reason to choose one over the other is convenience. In Newtonian mechanics, we prefer the Lagrangian approach rather than the Eulerian approach because we are concerned with trajectories of particles and also because the fluid in which the particles are embedded is not evident. The flow is in fact Hamiltonian, and if we don't perceive it in Newtonian mechanics it is because it is composed by all potential trajectories created by all the initial conditions that the system will allow. In the next chapter we will discuss in more detail this isomorphism between Hamiltonian and fluid mechanics.

Hamiltonian mechanics is based on the conservation of energy. Nevertheless, some systems are dissipative, and their energy decreases with the passing of time. (Increases in the energy would be a violation of the Second Law of Thermodynamics). The energy in fact doesn't disappear, it moves away from the system to an embedding system that contains it. So the appearance of loss of energy is due to us observing a small part of the whole. Mostly the energy becomes heat and it is measured as an increase in temperature on the original system. In fluids, the mechanisms by which this transfer of kinetic energy into heat occurs are friction and viscosity. Friction is the opposition of two surfaces to sliding over each other; work must be done to overcome friction and the energy associated with this work is dissipated as heat. Viscosity is a special type of friction that occurs between fluid particles, and it depends only on the velocity field shear. A third energy dissipation mechanism that is intrinsic only to three-dimensional fluids is turbulence. As we will see in later chapters, the vorticity field of a fluid obeys conservation laws expressed in Helmholtz's theorem. This theorem allows the creation of toroidal vortices that are impossible in less than three dimensions. Richardson proposed a mechanism known as the *energy cascade* in which larger vortices decay into smaller faster vortices all the way down to the molecular level, in which kinetic energy is transformed into heat. In a stroke of genius, Kolmogorov in 1941 was able to discover a power law that governs the energy cascade using only dimensional arguments (Davidson, 2004).

The important feature of the energy cascade in three dimensions is that vortices decay into smaller ones but the inverse effect is not likely: Smaller vortices cannot join to create a bigger one more often that they decay into smaller ones. This does not happen in two dimensions. Motivated by the Hamiltonian, in 1949 Onsager created a theory of point vortices in a two-dimensional fluid over a finite region. In this theory, he predicts negative temperature states. His argument was based on the fact that the physical space a vortex system occupies is the same as its phase space (making the Hamiltonian and fluid mechanics of the vortex system one and the same) (Onsager 1949 and Eyink, 2006). The core argument is that as we add more point vortices to a finite space, they have to spontaneously order themselves in order to not trample each other. Onsager's theory has been used by Chavanis (2002) to explain the spontaneous formation of vortical structures in stellar nebulae. The emergence of such vortices explains the formation of planets from the debris of a supernova.

Earth's atmosphere can be considered to be a two-dimensional fluid given that it is very thin, around 100 km, relative to the radius of the planet, 6,371 km. Onsager's negative temperature theory could explain why, while the atmosphere accumulates more energy, large vortical storms like hurricanes and typhoons are more frequent. Given that the preferred mechanism to dissipate energy for two-dimensional flows is friction, these large storms cannot dissipate their energy away until they make landfall.

Although Onsager's theory is very appealing and seems capable of explaining several phenomena, this thesis author and his advisor have been unable to verify it using computer modeling. Poje (2013) suspects its validity, and Berdichevsky et al (1991) think that Onsager's conclusions are a misapplication of the thermodynamic limit.

Fluid homogenization is the process by which a passive scalar mixes with a fluid. Picture a cup of coffee and add a single drop of cream. At the beginning, the drop of cream will have a smooth boundary, but if you leave it alone, small tendrils of cream will appear at the boundary moving into the coffee, and tendrils of coffee will start moving into the cream spot. After a long time (the coffee will be cold by then) both fluids will have perfectly mixed into each other forming a homogeneous liquid. This process is called diffusion and the tendrils are formed by molecules in Brownian motion intruding from one liquid into the other. The diffusion time is a function of the length scale of the system (the diameter of the cream spot) and the diffusion constant, depending on the liquids being mixed.

A second and faster way to obtain homogenization of the coffee-cream system is to stir it up. By stirring, adding vorticity to the system, the spot of cream is broken in smaller structures; the smaller structures have smaller length scales and, in consequence, smaller diffusion times. Hence, stirring the coffee-cream system homogenizes it faster. This process is called advection. The equation that governs the transport of a passive scalar (cream) in a fluid (coffee) is called the Advection-Diffusion equation. It is obtained from the Navier-Stokes equations for an incompressible flow and is non-linear.

In his Master's thesis, Chris Amey (2012) created a Hamiltonian analogue of two-dimensional fluids stirred by the Chirikov-Taylor Hamiltonian (Tabor, 1989), which is a very well known chaotic system. The objective of his research was to look for persistent patterns. Sundaram et al (2009) previously looked for persistent patterns and showed that there is multifractal structure with the same Hamiltonian. The method Amey employed consisted in expressing the diffusion of the passive scalar in frequency space, making the propagation operator a square matrix and finding the characteristic values. Amey and Sundaram discovered that the magnitude of the characteristic values follows a quadratic scaling rule. They speculated that there exists an analogy between their result and a popular quantum mechanical system—the particle in a box whose characteristic energies are also quadratically distributed. It is well known, but perhaps not well understood, that the Schrödinger equation is just the diffusion equation in imaginary time for which the role of the diffusion term is taken by the kinetic energy term. Sundaram speculated that the length of the box is the characteristic length of the island-like structures observed in the phase space of the Chirikov oscillator. Sundaram proposed to this author that we investigate this claim further by simplifying the shapes of the island, assuming them to be elliptical. The conclusion of this author is that the scaling rule observed by Amey cannot be explained only by diffusion. It has to be explained by the advection produced by a vorticity field. The effect of the advection is not evident because a feature of this system is that the gradient of the vorticity field is orthogonal to the stream function everywhere. However, absent this vorticity field, the characteristic functions of the system would show nodal lines that were not observed by Amey. Furthermore, the author claims that this orthogonal property is maintained for islands of non-elliptical shape by means of a conformal transformation. This thesis documents the results of this research.

Before closing this chapter, it should be mentioned that the Fluid-Hamiltonian mechanics analogy extends to non-relativistic quantum mechanics through Bohm's theory (Bohm, 1952). In 1951, David Bohm reformulated quantum mechanics by explicitly separating the phase from the magnitude of the wave function, creating two real fields. He found the equations of Hamilton-Jacobi and the equation of conservation of mass that are exact analogues of Hamiltonian and fluid mechanics, except for a new potential that he called *Quantum Potential*. Bohm's theory was not well accepted in his time because for some people, like Einstein, it was seen like a step back to classical mechanics. Einstein didn't like quantum mechanics in its post-Copenhagen interpretation shape. In fact, Einstein used Bohm's theory to criticize all of quantum mechanics and point out that something was wrong with the theory as a whole (Myrvold 2003). This early criticism and Bohm's own quasi-mystical interpretation turned people away from the theory. Nevertheless, J.S. Bell appears to hold Bohm's work in high regard and it was an inspiration for his work that produced Bell's inequality. Today, quantum chemists use Bohm's theory to calculate the trajectories of particles as small as molecules, transforming quantum mechanics de facto into fluid mechanics (Holland, 1997).

CHAPTER 2

SOME CONCEPTS OF FLUID MECHANICS

2.1 Eulerian and Lagrangian points of view

We mentioned in chapter 1 that there are two points of view to describe fluid kinematics, the Eulerian and the Lagrangian. Both are mathematically equivalent descriptions of fluid movement (see Childress 2009 and Kambe 2007).

The Eulerian point of view considers a continuous velocity field, $\vec{v}\left(\vec{x},t\right)$, defined in each point of the region of space and time that contains the fluid. The Lagrangian point of view considers the trajectory of each individual particle a inside the fluid described by the curve, $\vec{x}\left(a,t\right)$. The Eulerian and Lagrangian points of view are related by the set of differential equations:

$$\frac{d}{dt}\vec{x}\left(a,t\right)=\vec{v}\left(\vec{x}\left(a,t\right),t\right) \tag{2.1}$$

2.2 The convective derivative and some of its applications

The convective derivative or material derivative of a field is the operator:

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\vec{v}\left(\vec{x},t\right)\cdot\vec{\nabla} \tag{2.2}$$

Where $\vec{v}$ is an Eulerian velocity field. The convective derivative has properties similar to the conventional derivative. It is a linear operator that follows Leibniz rule:

$$\begin{aligned} \frac{D(f+g)}{Dt} &= \frac{Df}{Dt} + \frac{Dg}{Dt} \\ \frac{D(fg)}{Dt} &= \frac{Df}{Dt}g + f\frac{Dg}{Dt} \end{aligned} \tag{2.3}$$

Consider a scalar field that depends only in the spatial coordinate $\varphi\left(\vec{x}\right)$ and consider a time dependent translation of coordinates that coincides with the Lagrangian trajectories of the fluid, $\varphi\left(\vec{x} - \vec{x}\left(a,t\right)\right)$. Applying the convective derivative to the translated field:

$$\begin{aligned} &\frac{D}{Dt}\varphi\left(\vec{x} - \vec{x}\left(a,t\right)\right) = \left(\frac{\partial}{\partial t} + \vec{v}\left(\vec{x},t\right) \cdot \vec{\nabla}\right)\varphi\left(\vec{x} - \vec{x}\left(a,t\right)\right) = \\ &\vec{\nabla}\varphi\left(\vec{x} - \vec{x}\left(a,t\right)\right) \cdot \frac{d}{dt}\vec{x}\left(a,t\right) - \vec{v}\left(\vec{x},t\right) \cdot \vec{\nabla}\varphi\left(\vec{x} - \vec{x}\left(a,t\right)\right) = 0 \end{aligned} \tag{2.4}$$

The last equality follows (2.1). Hence the convective derivative can be thought of as a derivative in a coordinate system that moves with the fluid.

Two interesting cases of the convective derivative are constant velocity fields and rigid body rotations. For a constant velocity field $\vec{v}_0$ the corresponding solution of the Lagrangian equation is:

$$\vec{x} = \vec{x}_0 + \vec{v}_0 t \tag{2.5}$$

Following Kambe (2007) we can see equation (2.5) as a Galilean coordinate transform between the moving coordinate system $\vec{x}_0$ and the "fixed" coordinate system $\vec{x}$. We see that any particle distribution represented by φ will remain invariant from the point of view of a coordinate system moving at uniform speed $\vec{v}_0$.

The velocity field of a rigid body rotation around an axis through the origin is:

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{2.6}$$

Substituting in (2.2) and applying it to the position vector $\vec{r}$:

$$\frac{D\vec{r}}{Dt} = \frac{\partial \vec{r}}{\partial t} + (\vec{\omega} \times \vec{r}) \cdot \vec{\nabla} \vec{r} = \frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times \vec{r} \tag{2.7}$$

That is in fact the equation of the instantaneous speed of a body in a rotating frame (Fetter and Walecka 2003). Any distribution φ will be invariant from the point of view of the rotating coordinate system.

Another interesting application of the convective derivative is a one-dimensional Hamiltonian system for a particle of mass *m*, total energy *E* and potential energy $U(x)$. The velocity is expressed as:

$$v = \sqrt{\frac{2(E - U(x))}{m}} \tag{2.8}$$

Applying the convective derivative operator (2.2)

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 + \sqrt{\frac{2(E - U(x))}{m}} \frac{-\frac{2}{m}{\partial U}/{\partial x}}{2\sqrt{\frac{2(E-U(x))}{m}}} \tag{2.9}$$

From which we obtain Newton's Second Law:

$$\frac{D(mv)}{Dt} = -\frac{\partial U(x)}{\partial x} \tag{2.10}$$

As a final application we consider the classical action. Classical mechanics can be reformulated in terms of a field by means of the Hamilton-Jacobi equation (see Fetter 2003 and Arnold 1989) :

$$\frac{\partial S}{\partial t} + H\left(\vec{q}, \vec{\nabla} S, t\right) = 0 \tag{2.11}$$

Where *S* is the classical action, *H* is the Hamiltonian, $\vec{q}$ is a vector of positions of all the particles in the system and the momentum is:

$$\vec{p} = \vec{\nabla} S \tag{2.12}$$

Applying the convective derivative to the action we find:

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \frac{d\vec{q}}{dt} \cdot \vec{\nabla} S = -H\left(\vec{q}, \vec{p}, t\right) + \dot{\vec{q}} \cdot \vec{p} \tag{2.13}$$

Where we used (2.11) and (2.13) . The last expression is in fact the definition of the Lagrangian.

$$\frac{DS}{Dt} = L \tag{2.14}$$

2.3 Fluid dynamics concepts

The convective derivative allows formulating Newton's Second Law in terms of fields instead of discrete particles. In this sense:

$$\frac{D}{Dt} \left\{ \begin{array}{c} \text{Momentum} \\ \text{Density} \end{array} \right\} = \sum \{\text{Body Forces}\} \tag{2.15}$$

The momentum density is simply the mass density field times the speed field $\rho\vec{v}$. The body forces are more complex. In general, for a fluid we have three types of forces:

Pressure: It is a scalar field and reflects the thermodynamic effects in the movement of the fluid.

Viscosity: It is opposition of a fluid to shear stress.

External forces: Such as gravity or Lorentz forces.

When the fluid is incompressible and these three types of forces are acting on the fluid, equation (2.15) becomes the Navier-Stokes equations:

$$\frac{D\vec{v}}{Dt} = \vec{f} - \frac{\vec{\nabla} p}{\rho} + \eta\,\nabla^2 \vec{v} \tag{2.16}$$

Where p is the pressure, $\vec{f}$ are the body forces and η is the kinematic viscosity.

In the case in which the viscosity is absent, equations (2.16) are known as Euler equations and the fluid is called *inviscid*.

2.4 Conservation of mass

The statement that in any control volume, the net amount of fluid mass through the surface of the volume is equal to the net change in mass inside of it is expressed as:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot \rho\vec{v} = \frac{D\rho}{Dt} + \rho\,\vec{\nabla}\cdot\,\vec{v} = 0 \tag{2.17}$$

That formula is known as the Continuity Equation. When the flow in incompressible it reduces to:

$$\vec{\nabla}\cdot\vec{v} = 0 \tag{2.18}$$

sectionFluids in phase space, Liousville's theorem

The evolution of Hamiltonian systems of N particles in three-dimensional space is completely described by 6N coordinates corresponding to position and momentum of each particle and the Hamiltonian. Hamilton's equations are:

$$\begin{aligned} \dot{q}_i &= \tfrac{\partial H}{\partial p_i} \qquad i = 1.....3N \\ \dot{p}_i &= -\tfrac{\partial H}{\partial q_i} \end{aligned} \tag{2.19}$$

The fluid analogy is obtained by considering the system a fluid in 6N dimensional space.

With velocity vector of the form:

$$\vec{v} = \begin{bmatrix} \dot{\vec{q}} \\ \dot{\vec{p}} \end{bmatrix} \tag{2.20}$$

We see that the velocity vector (2.20) automatically honors (2.18) due to Hamilton's equations (2.19). In this case, for any function $\rho\left(\vec{q}, \vec{p}, t\right)$ that honors the continuity equation (2.17), we will have:

$$\frac{D\rho}{Dt} = 0 \tag{2.21}$$

This is Liousville's theorem.

2.5 Chapter conclusion

In this chapter we reviewed the convective derivative and showed some its applications. Expressing Newton's Second Law in terms of the convective derivative and considering the forces to which a fluid is submitted produced the Navier-Stokes equations. Furthermore, a system of particles in phase space can be considered an incompressible fluid and Louisville theorem follows from the fluid continuity equation.

CHAPTER 3

DIFFUSION INSIDE AN ELLIPTIC BOX

3.1 Problem Statement

The main objective of this chapter is to explain the scaling law Amey discovered regarding the characteristic values obtained from the spectral representation of the advection-diffusion equation. Figure 3.1 shows the quadratic law obtained from the spectral representation of the problem as explained in Amey's thesis (2012).

The approach of this chapter is to perturb a circular region into an ellipse by adding a slight eccentricity. Since the diffusion equation is for all intents and purposes the Schrödinger equation in imaginary time, we use the standard quantum mechanical perturbation method.

The diffusion equation is:

$$D\,\nabla^2\phi = \frac{\partial\phi}{\partial t} \tag{3.1}$$

with a Dirichlet type boundary condition, $\phi\left(x', y', t\right) = 0$. The boundary is an ellipse centered at the origin with semi-major axis, a, and eccentricity e. The boundary of the domain is,

$$\frac{x'^2}{a^2} + \frac{y'^2}{a^2\left(1-e^2\right)} = 1 \tag{3.2}$$

We can change the coordinates to transform the ellipse into the unit circle:

$$\begin{aligned} x' &= a\,x \\ y' &= a\sqrt{1-e^2}y \end{aligned} \tag{3.3}$$

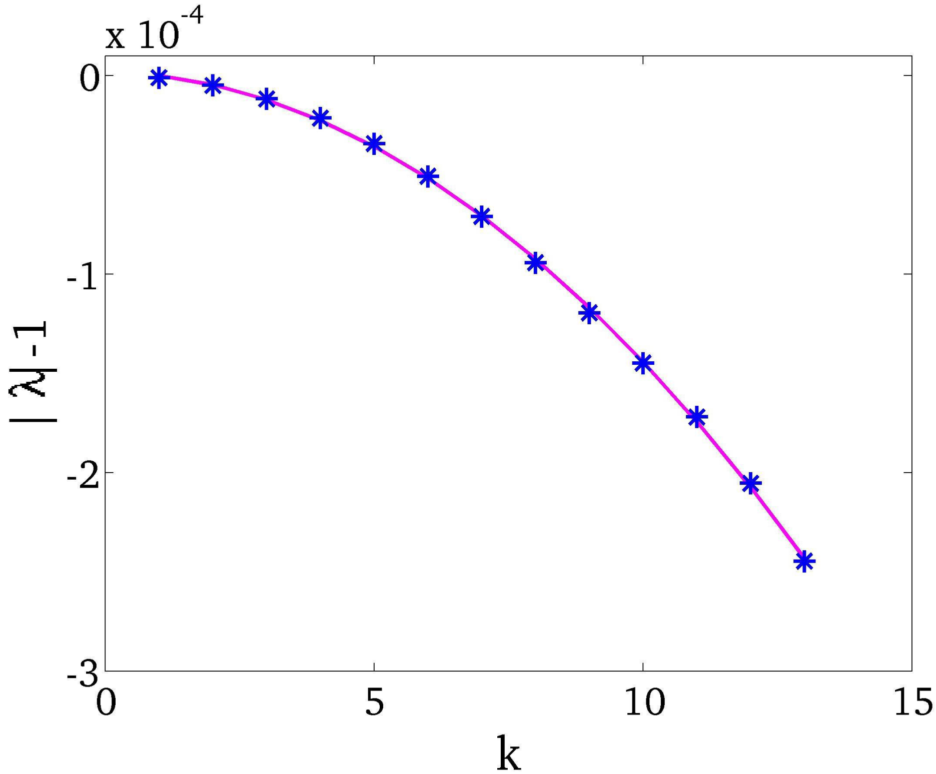

Figure 3.1: Quadratic dependence of the eigenvalue on the mode number as found by Amey (2012).

Then the diffusion equation (3.1) can be written in the new coordinate system:

$$\frac{D}{a^2}\frac{\partial^2\phi}{\partial x^2}+\frac{D}{a^2\left(1-e^2\right)}\frac{\partial^2\phi}{\partial y^2}=\frac{\partial\phi}{\partial t} \tag{3.4}$$

Consider solutions of the type $\phi\left(x,y,t\right)=\psi\left(x,y\right)e^{\frac{Et}{\hbar}}$. We can see that it will be required for the characteristic energy E to be negative in order for the solution to be stable as $t\rightarrow\infty$. With this type of solution, equation (3.4) becomes the characteristic equation:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{1}{\left(1-e^2\right)}\frac{\partial^2\psi}{\partial y^2}=\frac{a^2E}{D\hbar}\psi=\alpha\psi \tag{3.5}$$

3.2 Perturbation analysis

We apply a standard perturbation method. References can be found in Griffiths (2004) and Olshanii (2010). We can rewrite the characteristic equation (3.5) in the form:

$$H\psi+\varepsilon H_p\psi=\alpha\psi \tag{3.6}$$

With Hamiltonian:

$$H=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{3.7}$$

And perturbation Hamiltonian:

$$H_p=\frac{\partial^2}{\partial y^2} \tag{3.8}$$

The perturbation parameter:

$$\varepsilon=\frac{e^2}{1-e^2} \tag{3.9}$$

The unperturbed characteristic value problem can be written in polar coordinates as:

$$H\psi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\psi\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\psi=\alpha\psi \tag{3.10}$$

The boundary condition is $\psi\left(1,\theta\right)=0$, making ψ null on the unit circle boundary.

The partial differential equation (3.10) is well understood and can be solved by separation of variables. The characteristic functions are:

$$\psi_{n,k}^{(0)}\left(\rho,\theta\right)=\frac{1}{\sqrt{\pi}J_n'\left(\gamma_{n,k}\right)}J_n\left(\gamma_{n,k}\rho\right)e^{in\theta} \tag{3.11}$$

Where J_n is the Bessel function of the first kind of order *n*. $\gamma_{n,k}$ is the *k*-th zero of the *n*-th order Bessel function. Defining the inner product of two functions over the area of the unit circle as:

$$\left\langle a\right|\left.b\right\rangle=\int_{\text{Unit Circle}}a^*b\,dA=\int_0^{2\pi}\int_0^1 a^*b\,\rho\;d\rho\,d\theta \tag{3.12}$$

With this inner product we see that functions (3.11) form a complete set of orthogonal functions over the unit circle. In fact:

$$\left\langle\psi_{m,j}^{(0)}\right|\left.\psi_{n,k}^{(0)}\right\rangle=\delta_{m,n}\delta_{j,k} \tag{3.13}$$

The characteristic value is then:

$$H\psi_{m,k}=-\gamma_{n,k}^2\psi_{m,k} \tag{3.14}$$

And the corresponding characteristic energy:

$$E_{n,k}^{(0)}=-\gamma_{n,k}^2\frac{D\hbar}{a^2} \tag{3.15}$$

, which is negative as expected.

The first order perturbation solution of (3.6) is obtained by considering the perturbed characteristic function and characteristic value:

$$\psi_{n,k} = \psi_{n,k}^{(0)} + \varepsilon\, \psi_{n,k}^{(1)} + O\left(\varepsilon^2\right) \tag{3.16}$$

$$E_{n,k} = E_{n,k}^{(0)} + \varepsilon\, E_{n,k}^{(1)} + O\left(\varepsilon^2\right) \tag{3.17}$$

Using perturbation theory, the first order correction to the characteristic energy is:

$$E_{n,k}^{(1)} = \left\langle \psi_{n,k}^{(0)} \right| H_P \left| \psi_{n,k}^{(0)} \right\rangle \tag{3.18}$$

In order to calculate this value, we study first the action of the operator $\frac{\partial}{\partial y}$ on the characteristic functions. Expressing the operators in polar coordinates and applying Bessel identities (see mathematical appendix):

$$\frac{\partial}{\partial y}\psi_{n,k}^{(0)} = \frac{\gamma_{n,k} i}{2\sqrt{\pi} J_n'\left(\gamma_{n,k}\right)} \left(e^{i\theta(n+1)} J_{n+1}\left(\gamma_{n,k}\rho\right) + e^{i\theta(n-1)} J_{n-1}\left(\gamma_{n,k}\rho\right)\right) \tag{3.19}$$

And as a consequence:

$$H_P \psi_{n,k}^{(0)} = \frac{\partial^2}{\partial y^2}\psi_{n,k}^{(0)} = -\frac{\gamma_{n,k}^2}{4\sqrt{\pi} J_n'\left(\gamma_{n,k}\right)} \left(e^{i\theta(n+2)} J_{n+2}\left(\gamma_{n,k}\rho\right) + 2e^{in\theta} J_n\left(\gamma_{n,k}\rho\right) + e^{i\theta(n-2)} J_{n-2}\left(\gamma_{n,k}\rho\right)\right) \tag{3.20}$$

With this we can find the first order correction to the characteristic energy

$$\frac{a^2 E_{n,k}^{(1)}}{D\hbar} = \left\langle \psi_{n,k}^{(0)} \right| \frac{\partial^2}{\partial y^2} \left| \psi_{n,k}^{(0)} \right\rangle = -\frac{\gamma_{n,k}^2}{2} \tag{3.21}$$

And the corrected characteristic energy:

$$E_{n,k} = E_{n,k}^{(0)} + \varepsilon E_{n,k}^{(1)} = -\gamma_{n,k}^2 \frac{D\hbar}{a^2} + \left(\frac{e^2}{1-e^2}\right)\left(-\frac{\gamma_{n,k}^2}{2}\frac{D\hbar}{a^2}\right) = -\gamma_{n,k}^2 \frac{D\hbar}{2a^2}\left(\frac{2-e^2}{(1-e^2)}\right) \tag{3.22}$$

The first order correction to the characteristic functions is also obtained from perturbation theory:

$$\psi_{n,k}^{(1)} = \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} \sum_{j=1}^{\infty} \frac{\left\langle \psi_{m,j}^{(0)} \middle| H_P \middle| \psi_{n,k}^{(0)} \right\rangle}{\gamma_{m,j}^2 - \gamma_{n,k}^2} \psi_{m,j}^{(0)} \tag{3.23}$$

Knowing the value of the integral:

$$\int_0^{2\pi} \int_0^1 \rho J_m \left(\gamma_{m,j}\rho\right) e^{-im\theta} J_n \left(\omega\rho\right) e^{in\theta} d\rho d\theta = 2\pi\, \delta_{m,n} \frac{\gamma_{m,j} J_m \left(\omega\right) J_n' \left(\gamma_{m,j}\right)}{\omega^2 - \gamma_{m,j}^2} \tag{3.24}$$

, and working with expression (3.20) we can evaluate the numerators:

$$\left\langle \psi_{m,j}^{(0)} \middle| H_P \middle| \psi_{n,k}^{(0)} \right\rangle = \begin{cases} -\frac{\gamma_{n,k}^2}{2J_n'\left(\gamma_{n,k}\right)} \left(\frac{\gamma_{n-2,j} J_{n-2}\left(\gamma_{n,k}\right)}{\gamma_{n,k}^2 - \gamma_{n-2,j}^2} \right) & m = n-2 \\ -\frac{\gamma_{n,k}^2}{2J_n'\left(\gamma_{n,k}\right)} \left(\frac{\gamma_{n+2,j} J_{n+2}\left(\gamma_{n,k}\right)}{\gamma_{n,k}^2 - \gamma_{n+2,j}^2} \right) & m = n+2 \\ 0 & \text{Otherwise} \end{cases} \tag{3.25}$$

This produces the series:

$$\psi_{n,k}^{(1)} = \sum_{j=1}^{\infty} \frac{\left\langle \psi_{n-2,j}^{(0)} \middle| H_P \middle| \psi_{n,k}^{(0)} \right\rangle}{\gamma_{n-2,j}^2 - \gamma_{n,k}^2} \psi_{n-2,j}^{(0)} + \sum_{j=1}^{\infty} \frac{\left\langle \psi_{n+2,j}^{(0)} \middle| H_P \middle| \psi_{n,k}^{(0)} \right\rangle}{\gamma_{n+2,j}^2 - \gamma_{n,k}^2} \psi_{n+2,j}^{(0)} \tag{3.26}$$

And then the perturbation theory first order characteristic function is:

$$\psi_{n,k} = \psi_{n,k}^{(0)} - \left(\frac{e^2}{1-e^2} \right) \left(\sum_{j=1}^{\infty} \frac{\left\langle \psi_{n-2,j}^{(0)} \middle| H_P \middle| \psi_{n,k}^{(0)} \right\rangle}{\gamma_{n-2,j}^2 - \gamma_{n,k}^2} \psi_{n-2,j}^{(0)} + \sum_{j=1}^{\infty} \frac{\left\langle \psi_{n+2,j}^{(0)} \middle| H_P \middle| \psi_{n,k}^{(0)} \right\rangle}{\gamma_{n+2,j}^2 - \gamma_{n,k}^2} \psi_{n+2,j}^{(0)} \right) \tag{3.27}$$

3.3 Lowest mode analysis

The lowest mode corresponds to n=0 and the corresponding circular mode $\psi_{0,k}^{(0)}$ is independent of the azimuthal coordinate θ. Defining the perturbations coefficients as,

$$M_{k,j} = -\frac{\left\langle \psi_{-2,j}^{(0)} \middle| H_P \middle| \psi_{0,k}^{(0)} \right\rangle}{\gamma_{0,k}^2 - \gamma_{-2,j}^2} = -\frac{\left\langle \psi_{2,j}^{(0)} \middle| H_P \middle| \psi_{0,k}^{(0)} \right\rangle}{\gamma_{0,k}^2 - \gamma_{2,j}^2} = -\frac{\gamma_{0,k}^2}{2J_0'\left(\gamma_{0,k}\right)} \left(\frac{\gamma_{2,j} J_2\left(\gamma_{0,k}\right)}{\left(\gamma_{0,k}^2 - \gamma_{2,j}^2\right)^2} \right) \tag{3.28}$$

, we can write the lowest mode perturbed characteristic function:

$$\begin{aligned} \psi_{0,k} = \psi_{0,k}^{(0)} + \left(\tfrac{e^2}{1-e^2}\right) \left(\textstyle\sum_{j=1}^{\infty} M_{k,j} \frac{J_2(\gamma_{2,j}\rho)}{\sqrt{\pi} J_2'(\gamma_{2,j})}\right) \left(e^{i2\theta} + e^{-i2\theta}\right) = \\ \psi_{0,k}^{(0)} + 2\cos\left(2\theta\right) \left(\tfrac{e^2}{1-e^2}\right) \left(\textstyle\sum_{j=1}^{\infty} M_{k,j} \frac{J_2(\gamma_{2,j}\rho)}{\sqrt{\pi} J_2'(\gamma_{2,j})}\right) \end{aligned} \tag{3.29}$$

This is no longer independent of the azimuthal coordinate. In fact the perturbation term creates two nodal lines corresponding to $\theta = -{}^{\pi}/_{4}, {}^{\pi}/_{4}$. The nodal lines are an effect of order $O\left(e^2\right)$, which is very noticeable for high eccentricities. Figure 3.2 shows a plot of $\psi_{0,8}$ for e=0.95.

Nevertheless the results obtained by Amey do not show any nodal lines. An explanation for this is that the states with non-zero azimuthal modes $k > 0$ should not be part of the perturbation set.

The quantum mechanical angular momentum operator is:

$$\hat{l}_z = \begin{vmatrix} \hat{x} & \hat{y} \\ \frac{\hbar}{i}\frac{\partial}{\partial x} & \frac{\hbar}{i}\frac{\partial}{\partial y} \end{vmatrix} = \frac{\hbar}{i}\left(\hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial \theta} \tag{3.30}$$

The derivation of the last identity can be found on the appendix. Let's consider this operator on the elliptical coordinate system (3.3). Forcing the angular momentum to be zero for a

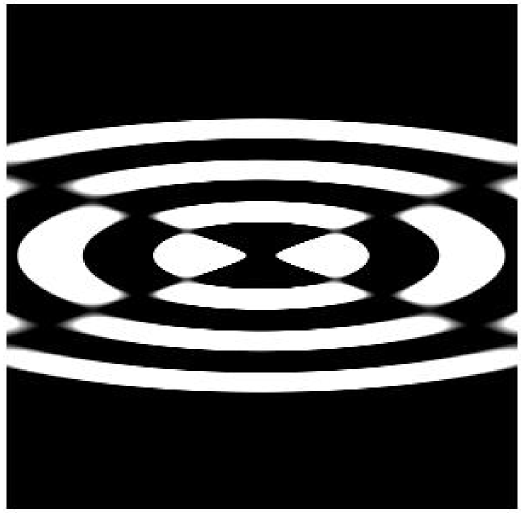

Figure 3.2: Nodal lines in perturbed eigenmode $\psi_{0,8}$ for eccentricity *e*=0.95.

function ϕ

$$\frac{\partial \phi}{\partial \theta} = -\left(\frac{1}{1-e^2}\right) y' \frac{\partial \phi}{\partial x'} + \left(1-e^2\right) x' \frac{\partial \phi}{\partial y'} = 0 \tag{3.31}$$

We can modify the original diffusion equation (3.1) by adding (3.31) multiplied by a constant $\Omega \neq 0$. What we obtain is the Advection-Diffusion equation:

$$D\frac{\partial^2 \phi}{\partial x'^2} + D\frac{\partial^2 \phi}{\partial y'^2} = \frac{\partial \phi}{\partial t} - \left(\frac{\Omega}{1-e^2}\right) y' \frac{\partial \phi}{\partial x'} + \Omega \left(1-e^2\right) x' \frac{\partial \phi}{\partial y'} \tag{3.32}$$

Or

$$D\frac{\partial^2 \phi}{\partial x'^2} + D\frac{\partial^2 \phi}{\partial y'^2} = \frac{\partial \phi}{\partial t} + \frac{\partial \eta}{\partial y'}\frac{\partial \phi}{\partial x'} - \frac{\partial \eta}{\partial x'}\frac{\partial \phi}{\partial y'} \tag{3.33}$$

In which η is the stream function:

$$\eta = -\frac{\Omega\left(1-e^2\right)}{2} x'^2 - \frac{\Omega}{2\left(1-e^2\right)} y'^2 + C \tag{3.34}$$

In other chapters we will discuss the relation of stream functions to Hamiltonians. In this case (3.34) corresponds to the Hamiltonian of a harmonic oscillator.

The perturbation of the diffusion equation from a circle into an ellipse forced us to include non-azimuthally symmetric modes and, in consequence, obtain non-azimuthally symmetric perturbed characteristic functions. We argue now that if we study instead the advection-diffusion equation with stream function (3.34) the only modes that should be considered are the azimuthally symmetrical ones. For this purpose we prove the following theorem.

Theorem 1. *Consider the characteristic equation of* (3.32) *with a unit circle boundary and null eccentricity* $e = 0$*. If* $\Omega \neq 0$ *then the only characteristic functions of the complete orthogonal set* $J_n(\alpha\rho)\, e^{in\theta}$ *that it admits as solutions are those with zero angular momentum.*

Proof. Writing the characteristic equation with null eccentricity in polar coordinates, we get:

$$\left(D\nabla^2 - \Omega \frac{\partial}{\partial\theta} \right) \psi = \varepsilon\psi \tag{3.35}$$

The eigenfunctions of the complete orthogonal set for the two dimensional Laplace operator in polar coordinates are of the form

$$\psi_n = Re\left[(c_1 - c_2 i)\ J_n(\alpha\rho)\, e^{in\theta} \right] = J_n(\alpha\rho)\,(c_1 \cos n\theta + c_2 \sin n\theta) \tag{3.36}$$

for $n = -\infty \ldots 0 \ldots \infty$. Substituting in (3.35):

$$\begin{aligned}
&\left(D\nabla^2 - \Omega\tfrac{\partial}{\partial\theta}\right) J_n\left(\alpha\rho\right)\left(c_1\cos n\theta + c_2\sin n\theta\right) = \\
&-\alpha^2 D J_n\left(\alpha\rho\right)\left(c_1\cos n\theta + c_2\sin n\theta\right) - n\,\Omega\, J_n\left(\alpha\rho\right)\left(-c_1\sin n\theta + c_2\cos n\theta\right) = \\
&\left(-c_1\alpha^2 D - c_2 n\Omega\right) J_n\left(\alpha\rho\right)\cos n\theta + \left(c_1 n\,\Omega c_1 - \alpha^2 D c_2\right) J_n\left(\alpha\rho\right)\sin n\theta = \\
&= \varepsilon c_1 J_n\left(\alpha\rho\right)\cos n\theta + \varepsilon c_2 J_n\left(\alpha\rho\right)\sin n\theta
\end{aligned} \tag{3.37}$$

This produces the eigenvalue problem:

$$\begin{bmatrix} -\alpha^2 D & -n\,\Omega \\ n\,\Omega & -\alpha^2 D \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \varepsilon \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{3.38}$$

The associated characteristic equation is:

$$\left(\varepsilon + \alpha^2 D\right)^2 + n^2\Omega^2 = 0 \tag{3.39}$$

With roots

$$\varepsilon = -\alpha^2 D \pm n\,\Omega\, i \tag{3.40}$$

Now for reasons explained in section 3.1, ε is a physical quantity that is proportional to the characteristic energies of the system and therefore should be a negative real number, for this to be true either *n* or Ω should be zero. Since we established Ω is a real number different from zero, *n* is zero. □

The theorem proves that once we consider the advection caused by the stream function (3.34) the modes $k > 0$ are no longer part of the set of characteristic functions and there is no need to consider them in the perturbation process. With this we find that the characteristic functions are of the form:

$$\psi_{0,k} = J_0 \left(\gamma_{0,k} \sqrt{x^2 + \frac{y^2}{1-e^2}} \right) \tag{3.41}$$

We can see that for the circle *e*=0, the advection term of (3.32) is always zero. This disguises the advection in the equation making it look as a diffusion only equation.

3.4 Chapter conclusion

In this chapter we studied the diffusion inside an elliptic region through a first order perturbation from a circle. We showed that the resultant characteristic functions have nodal lines and are not azimuthally symmetrical. Since such nodal lines were not observed by Amey, we concluded that the advection term was missing from the original equation (3.1). Once we included it by considering a quadratic stream function (3.34) , we found that the system only admitted azimuthally symmetric characteristic functions. With this the nodal line states disappeared and the perturbed characteristic function is (3.41). The characteristic energies are:

$$E_{0,k} = -\gamma_{0,k}^2 \frac{D\hbar}{2a^2} \left(\frac{2-e^2}{(1-e^2)} \right) \tag{3.42}$$

Since the zeros of the Bessel functions are asymptotically distributed as *k*. We find from the last expression the scaling rule:

$$E_{0,k} \sim -\hbar D \frac{k^2}{b^2} + O\left(e^2\right) \tag{3.43}$$

Where b is the minor axis of the ellipse, which is in fact the characteristic length of the system. Expression (3.43) explains Amey's findings.

CHAPTER 4

VORTICITY, DIFFUSION AND CONFORMAL TRANSFORMATIONS

4.1 Two-dimensional vorticity

Consider the Navier-Stokes equations for an incompressible fluid:

$$\begin{aligned} &\tfrac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot \vec{\nabla}\right) \vec{v} = -\tfrac{1}{\rho}\vec{\nabla} p + D\, \nabla^2 \vec{v} \\ &\vec{\nabla}\cdot \vec{v} = 0 \end{aligned} \tag{4.1}$$

The two-dimensional version of these equations is obtained by assuming that the velocity component in the z direction is constant $v_z = \mathrm{const.}$ and the pressure p is not a function of z. Applying the curl operator to (4.1) and defining the vorticity field for a two-dimensional fluid as $\omega = -\vec{e}_z\cdot \left(\vec{\nabla} \times \vec{v}\right)$, the Navier-Stokes equations reduce to:

$$\frac{\partial \omega}{\partial t} + v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} = D \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{4.2}$$

Remembering that the velocities are obtained from the stream function ψ as $\vec{v} = -\vec{e}_z \times \vec{\nabla}\psi$ we obtain the vorticity transport equation:

$$\frac{\partial \omega}{\partial t} + \{\omega, \psi\} = D \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{4.3}$$

That is in fact the advection-diffusion equation when considering ω as the passive scalar. Notice that the advection term is the Poisson bracket of the vorticity field with the stream function. However vorticity is far from not passive it is the engine that keeps the fluid moving. In fact:

$$\omega = -\vec{e}_z \cdot \left(\vec{\nabla} \times \vec{v}\right) = -\vec{e}_z \cdot \left(\vec{\nabla} \times \left(-\frac{\partial \psi}{\partial y}\vec{e}_x + \frac{\partial \psi}{\partial x}\vec{e}_y\right)\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{4.4}$$

Together (4.3) and (4.4) form a system of two partial differential equations in two variables.

4.2 Kinetic energy of the fluid

The kinetic energy of the fluid mass in a region bounded by a stream line is:

$$KE = \frac{\rho}{2}\int_R \left(\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right) dxdy \tag{4.5}$$

Integrating by parts and considering that in the boundary the stream function goes to zero:

$$KE = -\frac{\rho}{2}\int_R \omega\, \psi dx\, dy \tag{4.6}$$

4.3 Azimuthally symmetric solutions

Let's consider a case in which the stream function and the vorticity fields depend only of the distance to the origin. The advection term becomes (see mathematical appendix):

$$\{\omega, \psi\} = \frac{\partial \omega}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y}\frac{\partial \psi}{\partial x} = \left(\cos\theta \frac{\partial \omega}{\partial \rho}\right)\left(\sin\theta \frac{\partial \psi}{\partial y}\right) - \left(\sin\theta \frac{\partial \omega}{\partial \rho}\right)\left(\cos\theta \frac{\partial \psi}{\partial y}\right) = 0 \tag{4.7}$$

As a consequence the azimuthally symmetric fields produce no advection. This makes equation (4.3) a pure diffusion equation that can be solved for the vorticity ω. Then the stream function can be found from equation (4.4) which is just the two-dimensional Poisson equation. Its general solution is found from Biot-Savart's law (see Wayne 2011). We now discuss a few examples.

4.3.1 Constant vorticity

Let the vorticity be a constant real number over the fluid. It trivially honors equation (4.3). Equation (4.4) becomes:

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)\right) = \omega \tag{4.8}$$

That integrates into:

$$\psi = \frac{\omega\, r^2}{4} + \psi_H \tag{4.9}$$

Selecting $\psi_H = -\frac{\omega\, R^2}{4}$ in order to make the stream function vanish at a boundary of radius *R*.

4.3.2 Point vortex at the origin

In this case, the vorticity is a pulse at the origin:

$$\omega = \frac{\Gamma}{4\pi}\delta\left(r\right) \tag{4.10}$$

The stream function is the Green's function for the Poisson equation:

$$\psi\left(r\right) = \frac{\Gamma}{2\pi}\ln|r| \tag{4.11}$$

A vortex system can be created by linearly combining point vortices in the same fashion as electric charges are combined to create a more complex electric field. (see Newton 2001 and Saffman 1992).

4.3.3 Lamb-Oseen Vortex

Solving the diffusion equation by itself under the condition that the vorticity vanishes at infinity, produces:

$$\omega = \frac{\Gamma}{4\pi D\,t} \exp\left(-\frac{r^2}{4D\,t}\right) \tag{4.12}$$

The constant Γ is known as the circulation, $r = \sqrt{x^2 + y^2}$ is the distance to the origin and $2\sqrt{D\,t}$ is known as the vortex core radius. The vorticity can be expressed in terms of the Hankel transform (Arfken 2011):

$$\omega = \frac{\Gamma}{2\pi} \int_0^\infty e^{-D\,tk'^2} J_0\left(k'\,r\right) k'\,dk' = \frac{\Gamma}{2\pi\nu t} \int_0^\infty e^{-k^2} J_0\left(\frac{k\,r}{\sqrt{D\,t}}\right) k\,dk \tag{4.13}$$

Since Bessel functions of the first kind are solutions to the Helmholtz equation,

$$\nabla^2 J_0\left(k\,r\right) + k^2 J_0\left(k\,r\right) = 0 \tag{4.14}$$

we can solve for the stream function from (4.4), obtaining:

$$\psi = -\frac{\Gamma}{2\pi} \int_0^\infty \frac{e^{-k^2}}{k} J_0\left(\frac{k\,r}{\sqrt{D\,t}}\right) dk + \psi_H \tag{4.15}$$

, where the last term is the homogeneous solution of the Poisson equation. We can see that the streamlines are circles with center at the origin and radius that grows proportionally to $\sqrt{D\,t}$.

4.3.4 Vortical diffusion in a region bounded by a unit circle

Consider a boundary that coincides with the unit circle centered at the origin. The vorticity and the stream function vanish at the boundary. By arguments similar to section 3.2, both the vorticity and stream function are linear combinations of the characteristic functions obtained by solving the Helmholtz equation on a unit circle boundary. With this consideration the vorticity field is:

$$\omega = \sum_{k=1}^{\infty} c_k e^{-\gamma_{0,k}^2 D\, t} J_0\left(\gamma_{0,k} r\right) \tag{4.16}$$

where $c_k,\ k = 1, ..., \infty$ are real numbers. From the properties of the Bessel functions of the first kind we can obtain the corresponding stream function:

$$\psi = -\sum_{k=1}^{\infty} c_k \frac{e^{-\gamma_{0,k}^2 D\, t}}{\gamma_{0,k}^2} J_0\left(\gamma_{0,k} r\right) \tag{4.17}$$

4.4 Conformal mapping

4.4.1 Transforming the Advection-Diffusion-Kinematics equation pair

In the last section we saw that if the stream function and the vorticity field depend only of the distance to the origin the advection term of (4.3) is zero. We can interpret geometrically this condition as the velocity field being perpendicular to the vorticity gradient. The need to maintain azimuthal symmetry forces the boundary to always be circular; however we can use a Conformal Mapping to transform the boundary into other shapes. Since by definition a Conformal Mapping does not change internal angles of a region of space, a new vorticity stream pair obtained this way will maintain orthogonality. Therefore the advection term will continue to be zero and the equation system will still be separable.

As shown in the mathematical appendix the Advection-Diffusion and Kinematic equation pair (4.3) and (4.4) can be expressed in complex coordinates, $z = x + iy,\ \bar{z} = x - iy$ as:

$$\begin{aligned} &\tfrac{\partial \omega_z}{\partial t} + Im\left[\bar{\nabla}\psi\, \nabla\omega_z\right] = D\, \nabla_z^2 \omega_z \\ &\nabla_z^2 \psi = \omega_z \end{aligned} \tag{4.18}$$

Following Bazant (2004) let us define the conformal transformation by the holomorphic function $\zeta = w(z) = \varepsilon(x,y) + i\,\eta(x,y)$ (Mei 1997). The equation pair (4.18) takes the form:

$$\begin{aligned}&\tfrac{\partial \omega_z}{\partial t} + 4\left(\tfrac{dw}{dz}\right)^* \left(\tfrac{dw}{dz}\right) Im\left[\tfrac{\partial \psi}{\partial w}\tfrac{\partial \omega_z}{\partial \bar{w}}\right] = D\left(\tfrac{dw}{dz}\right)^* \left(\tfrac{dw}{dz}\right)\nabla_\zeta^2 \omega_z \\ &\left(\tfrac{dw}{dz}\right)^* \left(\tfrac{dw}{dz}\right)\nabla_\zeta^2 \psi = \omega_z\end{aligned} \tag{4.19}$$

The vorticity ω_ζ in the new coordinate system is defined as $\omega_\zeta = \nabla_\zeta^2 \psi$. The relationship of vorticity in both systems is thus:

$$\omega_z = \left|\frac{dw}{dz}\right|^2 \omega_\zeta \tag{4.20}$$

Giving us the equation pair in the new coordinates as:

$$\begin{aligned}&\tfrac{\partial \omega_\zeta}{\partial t} + Im\left[\bar{\nabla}_\zeta \psi\, \nabla_\zeta \omega_\xi\right] = D\,\nabla_\zeta^2 \omega_\zeta \\ &\nabla_\zeta^2 \psi = \omega_\zeta\end{aligned} \tag{4.21}$$

Focusing on the advection term, we can see that (mathematical appendix):

$$\{\psi, \omega\}_{(x,y)} = \left|\frac{dw}{dz}\right|^2 \{\psi, \omega\}_{(\varepsilon,\eta)} \tag{4.22}$$

This property applied to systems with azimuthal symmetry means that if (4.18) is separable; it will continue to be separable in all new coordinate systems that are obtained through a conformal transformation. To see this, consider the real functions $\psi(r^2) = \psi(z\bar{z})$ and $\omega(r^2) = \omega(z\bar{z})$, applying the Poisson bracket operation to them:

$$\begin{aligned}&\{\psi, \omega\}_{(x,y)} = 4Im\left[\tfrac{\partial \psi}{\partial z}\tfrac{\partial \omega}{\partial \bar{z}}\right] = 4Im\left[\left(\psi'(z\bar{z})\,\bar{z}\right)\left(\omega'(z\bar{z})\,z\right)\right] = \\ &4Im\left[z\bar{z}\psi'(z\bar{z})\,\omega'(z\bar{z})\right] = 0\end{aligned} \tag{4.23}$$

Then advection is null everywhere in the transformed region where $\left|\frac{dw}{dz}\right|^2 > 0$, thus proving that the vorticity can be solved by itself.

4.4.2 Linear coordinate transformations

A special case of a conformal transformation is:

$$w\left(z\right)=\lambda e^{i\theta}\left(z+z_{0}\right) \tag{4.24}$$

The linear transformation produces three effects applied in the following order:

Translation, the origin is shifted to z_0.

Rotation, the coordinate axes are rotated counterclockwise and angle θ.

Dilation, the axes scale an amount λ.

Translation and rotation affect only the spatial coordinates. If, in addition, we scale time $s=\lambda^2 t$, the Advection-Diffusion-Kinematics equation pair turns out to be invariant:

$$\begin{aligned}&\tfrac{\partial\omega_\zeta}{\partial s}+Im\left[\bar{\nabla}_\zeta\psi\,\nabla_\zeta\omega_\xi\right]\;=D\,\nabla_\zeta^2\omega_\zeta\\&\nabla_\zeta^2\psi=\omega_\zeta\end{aligned} \tag{4.25}$$

It is well understood in literature that the diffusion equation is invariant to scaling of time and characteristic length. We showed here that it is also true for (4.18). Also notice from (4.20), that vorticity scales as λ^2, since its measure is in units reciprocal of time.

Another way to approach linear transformations is to absorb the scaling factor into the diffusion coefficient obtaining:

$$D_\xi=\lambda^2 D_z \tag{4.26}$$

4.5 Elliptical regions

4.5.1 Jouwkovsky transformation

Consider a special case of the Jouwkovsky transformation (Milne-Thomson 1958):

$$z(\xi) = \frac{a+b}{2}\xi + \frac{a-b}{2\xi} \tag{4.27}$$

Where a and b are real numbers such that $a > b > 0$. The transformation (4.27) maps the unit circle in the complex plane of ξ into an ellipse with semi-major axis of length a that coincides with the $Re[\xi]$ axis and semi-minor axis of length b. This is easy to see by substituting $\xi = e^{i\theta}$ into (4.27), obtaining:

$$z(\xi) = a\,\cos\theta + i\,b\,\sin\theta \tag{4.28}$$

That is the parametric equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Differentiating (4.27), we find:

$$\frac{dz}{d\xi} = \frac{a+b}{2} - \frac{a-b}{2\xi^2} \tag{4.29}$$

That is identical to zero in the circle:

$$\xi = \sqrt{\frac{a-b}{a+b}}e^{i\theta} \tag{4.30}$$

Substituting back (4.30) in (4.27), we find $z = \sqrt{a^2 - b^2}\cos\theta$, that is the line joining the foci of the ellipse. This line is the branch cut of the transformation. The two branches are obtained by solving ξ in (4.27).

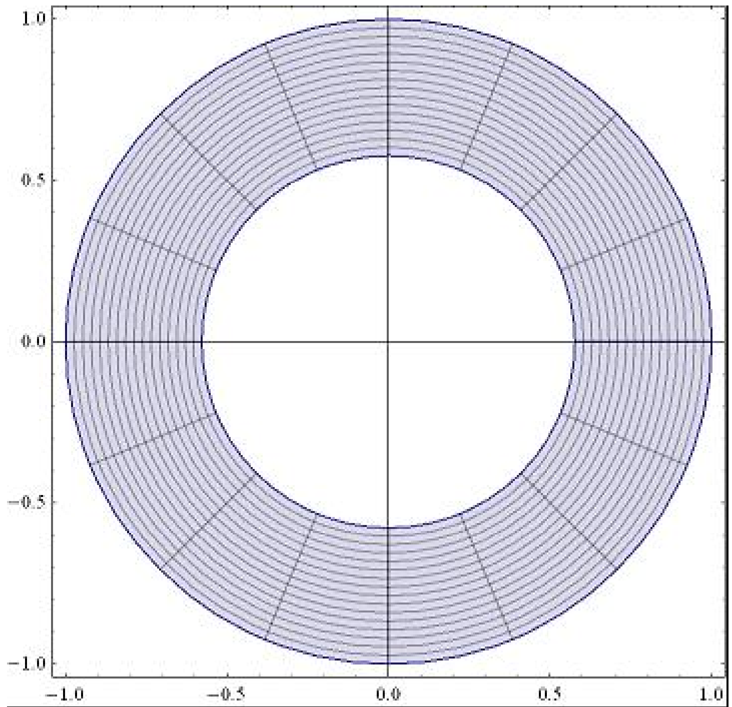

Figure 4.1: Domain of the Jouwkovsky transformation of the ellipse. The principal branch excludes the circle at the origin with radius $\sqrt{\frac{a-b}{a+b}}$.

Principal branch: This branch corresponds to the inverse transformation:

$$\xi = \frac{z + \sqrt{z^2 - c^2}}{a + b} \tag{4.31}$$

Where $c = \sqrt{a^2 - b^2}$ is the distance of the foci from the origin. The interior of the ellipse in this branch maps to the interior of a ring limited by circles of radius $\sqrt{\frac{a-b}{a+b}}$ and 1. Figures 4.1 and 4.2 represent the ring in the ξ and the ellipse in the z plane respectively. In this branch, $\xi \to \infty$ as$z \to \infty$.

Secondary Branch: This branch is represented by the transformation:

$$\xi = \frac{z - \sqrt{z^2 - c^2}}{a + b} \tag{4.32}$$

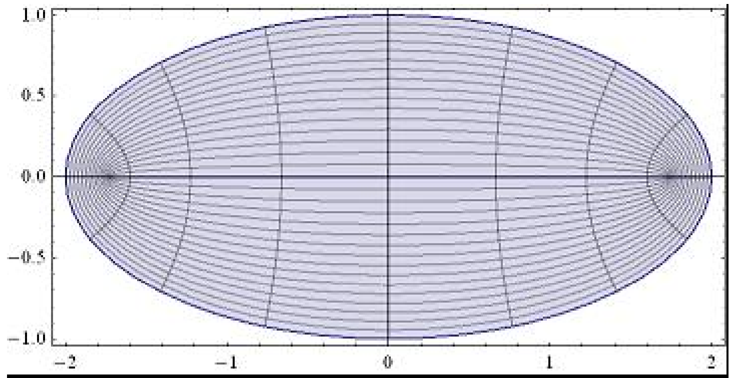

Figure 4.2: Image of the Jouwkovsky transformation. The unit circle is the perimeter of the ellipse.

In this branch the perimeter of the ellipse is mapped to the circle $\frac{a-b}{a+b}e^{i\theta}$, that defines the inner radius of a ring in the ξ plane, which maps the interior of the ellipse. The ring is represented in figure 4.3. In this branch $\xi \to 0$ as $z \to \infty$.

4.5.2 Elliptical regions with small eccentricity

Let's consider regions that are elliptical with slight eccentricity. The original region is an ellipse on the complex plane centered at the origin with semi-major axis length a, semi-minor axis length b and coordinates $z = x + i\,y$. Using (4.31) we project the original region to the plane $\xi = \varepsilon + i\eta$. We calculate:

$$\frac{d\xi}{dz} = \frac{1 + {}^{z}\!/\!\sqrt{z^2 - a^2e^2}}{a+b} \tag{4.33}$$

For small eccentricities$\left|{}^{z}\!/_{a}\right| >> e$,

$$\frac{d\xi}{dz} = \frac{2}{a+b} \tag{4.34}$$

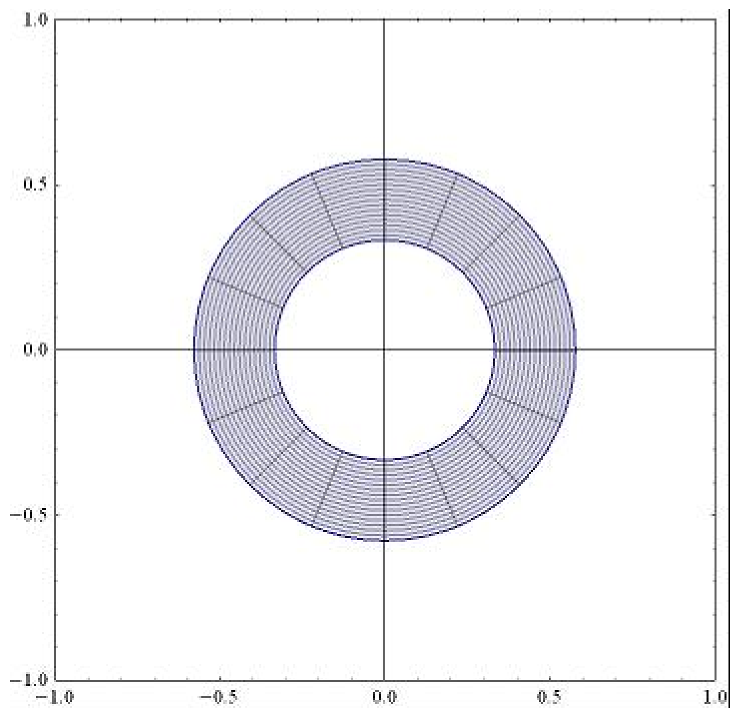

Figure 4.3: Secondary branch of the Jouwkovsky transformation. The ellipse is mapped to the ring between $\frac{a-b}{a+b}$ and $\sqrt{\frac{a-b}{a+b}}$.

Under this expression, the Advection-Diffusion equation (4.21) takes the form:

$$\frac{\partial \omega_\zeta}{\partial t} = \frac{4D_{\text{Ellipse}}}{(a+b)^2} \nabla_\zeta^2 \omega_\zeta \tag{4.35}$$

This shows that the effect of mapping a slightly elliptical region into the unit circle results in a change in the diffusion coefficient:

$$D_{\text{Unit Circle}} = \frac{4D_{\text{Ellipse}}}{(a+b)^2} \tag{4.36}$$

Using the results of Chapter 3, we find that the characteristic energies are:

$$E_{n,0} = -\gamma_{n,0}^2 \hbar D_{\text{Unit Circle}} = -\gamma_{n,0}^2 \hbar \frac{4D_{\text{Ellipse}}}{(a+b)^2} \tag{4.37}$$

Or

$$E_{n,0} = -\gamma_{n,0}^2 \hbar \frac{D_{\text{Ellipse}}}{a^2} + O\left(e^2\right) \tag{4.38}$$

That is a result similar to the one obtained in Chapter 3 through perturbation methods.

4.6 Chapter conclusion

We showed in this chapter that the two-dimensional Navier-Stokes equations for an incompressible fluid can be rewritten in terms of two scalar fields, vorticity and stream function. Additionally, we showed that one of the equations obtained is the Advection-Diffusion equation. If the gradient of the vorticity is orthogonal to the tangent of the stream function, the advection term is null and the vorticity is the solution of the diffusion equation only, making the system separable. One way to obtain this orthogonality is by making the vorticity and stream fields depend only of the distance to origin. This makes all stream lines circles centered at the origin. We showed that we can transform the boundary into other shapes by using conformal mappings and obtained a new pair of equations (4.21) in the transformed coordinates.

We saw that linear transformations (4.24) are special in the sense that they preserve the structure of the Advection-Diffusion-Kinetic equation pair. The effect of the transformation is a rescaling of the characteristic time that can be absorbed into the diffusion coefficient.

Finally, we applied the theory to an elliptical region with a slight eccentricity. We found that in this case the conformal mapping is essentially a scaling transformation. This result confirms the characteristic energies obtained in Chapter 3.

CHAPTER 5

SYMPLECTIC CONSIDERATIONS

5.1 Symplectic group

Consider the 2n by 2n square matrix:

$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \tag{5.1}$$

where I_n is the identity matrix. J_n is known as the *symplectic* matrix. The set of invertible square matrices A with complex entries such that

$$A^{\dagger} J_n A = J_n \tag{5.2}$$

forms a group. This group is known as the **symplectic group** Sp(n) (Stillwell 2008). It is easy to see that for all the members of the group $|\det(A)| = 1$.

5.2 Canonical transformations of Hamilton's equations

Consider the state of a system of M particles described by the position vector $\vec{q}$ and the momentum vector $\vec{p}$. The state vectors dimension is N=(Dimensions of embedding space)xM numbers. Hamilton's equations are:

$$\frac{d}{dt}\begin{bmatrix}\vec{q}\\ \vec{p}\end{bmatrix} = J_N \begin{bmatrix}\vec{\nabla}_{\vec{q}} H\\ \vec{\nabla}_{\vec{p}} H\end{bmatrix} \tag{5.3}$$

Consider a transformation of coordinates such as $\vec{q'} = \vec{q'}\left(\vec{q},\vec{p}\right)$ and $\vec{p'} = \vec{p'}\left(\vec{q},\vec{p}\right)$with Jacobian $\frac{\partial\left(\vec{q'},\vec{p'}\right)}{\partial\left(\vec{q},\vec{p}\right)}$, then Hamilton's equations take the form:

$$\frac{d}{dt}\begin{bmatrix}\vec{q'}\\ \vec{p'}\end{bmatrix} = \frac{\partial\left(\vec{q'},\vec{p'}\right)}{\partial\left(\vec{q},\vec{p}\right)}^{\dagger} J_N \frac{\partial\left(\vec{q'},\vec{p'}\right)}{\partial\left(\vec{q},\vec{p}\right)} \begin{bmatrix}\vec{\nabla}_{\vec{q'}} H\\ \vec{\nabla}_{\vec{p'}} H\end{bmatrix} \tag{5.4}$$

For the transformation to be Canonical, it has to preserve the form of Hamilton's equations. Therefore:

$$\frac{\partial\left(\vec{q'},\vec{p'}\right)}{\partial\left(\vec{q},\vec{p}\right)}^{\dagger} J_N \frac{\partial\left(\vec{q'},\vec{p'}\right)}{\partial\left(\vec{q},\vec{p}\right)} = J_N \tag{5.5}$$

The Jacobian of a Canonical transformation is an element of Sp(n).

5.3 Poisson's brackets and Liouville's theorem

The Poisson Bracket defined in terms of the symplectic matrix is:

$$\{f,g\} = \begin{bmatrix}\vec{\nabla}_{\vec{q}} f\\ \vec{\nabla}_{\vec{p}} f\end{bmatrix}^{\dagger} J_n \begin{bmatrix}\vec{\nabla}_{\vec{q}} g\\ \vec{\nabla}_{\vec{p}} g\end{bmatrix}^{\dagger} \tag{5.6}$$

From (5.3), and considering that f is a function of position, momentum and time:

$$\{f,H\} = \begin{bmatrix}\vec{\nabla}_{\vec{q}} f\\ \vec{\nabla}_{\vec{p}} f\end{bmatrix}^{\dagger} \frac{d}{dt}\begin{bmatrix}\vec{q}\\ \vec{p}\end{bmatrix} = \frac{df}{dt} - \frac{\partial f}{\partial t} \tag{5.7}$$

As a consequence, Liouville's theorem tells us that if ρ is a conserved quantity then we have:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \{\rho, H\} = 0 \tag{5.8}$$

5.4 Conformal transformations of stream functions

Stream functions in the Lagrangian context have the same role as Hamilton's equations for particles. Using the symplectic group we will study under which condition conformal transformations are canonical.

By definition of stream function for a two-dimensional flow (we refer to J_1 as J):

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = J\,\vec{\nabla}_{(x,y)}\psi \tag{5.9}$$

In order for the conformal transformation $x = x(\varepsilon, \eta)$, $y = y(\varepsilon, \eta)$ to be canonical, it's Jacobian has to be part of the symplectic group Sp(1). That is:

$$\frac{\partial(\varepsilon, \eta)}{\partial(x, y)}^{\dagger} J \frac{\partial(\varepsilon, \eta)}{\partial(x, y)} = J \tag{5.10}$$

Carrying away the matrix operation and applying the Cauchy-Riemann conditions, we find that the condition for a transformation to be both canonical and conformal is:

$$\left(\frac{\partial \varepsilon}{\partial x}\right)^2 + \left(\frac{\partial \varepsilon}{\partial y}\right)^2 = \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 = 1 \tag{5.11}$$

Isometries, the combinations of rotations and translations through $w(z) = e^{i\theta}(z - z_0)$, are transformations that are both canonical and conformal. Dilation is not, although as explained in Chapter 4 we could re-scale time to obtain a new set of Hamilton-like equations. We will take a different approach in this chapter.

5.5 Transformation of time dependent stream functions

Let's analyze now a stream function of the form $\psi(x, y, t)$. Following Morrison (1998) we built the time independent Hamiltonian:

$$H = \psi(x, y, \tau) + \phi \tag{5.12}$$

Where we added two new conjugated variables τ and ϕ . The corresponding Hamilton equations for the new variables are:

$$\begin{aligned} \tfrac{d\tau}{dt} &= \tfrac{\partial H}{\partial \phi} = 1 \\ \tfrac{d\phi}{dt} &= -\tfrac{\partial \psi}{\partial \tau} \end{aligned} \tag{5.13}$$

We see from the equations for the new variable that $\tau = t$, making time a Hamiltonian variable. Let's call the Jacobian of a conformal transformation on space variables *x* and *y*, $C = \frac{\partial(\varepsilon,\eta)}{\partial(x,y)}$. Also let's rearrange the symplectic matrix in the following way:

$$J_2 = \begin{bmatrix} J_1 & 0 \\ 0 & J_1 \end{bmatrix} \tag{5.14}$$

We would like to construct a symplectic transformation for the Hamiltonian (5.12) that contains a conformal transformation on the space variables as part of it. The transformation matrix has the shape:

$$\Lambda = \begin{bmatrix} C & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{5.15}$$

Where X_{12}, X_{21} and X_{22} are unknown 2x2 matrices. To be canonical, Λ has to meet the condition $\Lambda^{\dagger} J_2 \Lambda = J_2$, which produces the following system of equations:

$$\begin{aligned} C^{\dagger} J_1 C + X_{21}^{\dagger} J_1 X_{21} &= J_1 \\ X_{12}^{\dagger} J_1 C + X_{22}^{\dagger} J_1 X_{21} &= 0 \\ X_{12}^{\dagger} J_1 X_{12} + X_{22}^{\dagger} J_1 X_{22} &= J_1 \end{aligned} \tag{5.16}$$

By (5.11) we know that:

$$C^{\dagger} J_1 C = \omega^2 J_1 \tag{5.17}$$

Where $\omega^2 = \left(\frac{\partial \varepsilon}{\partial x}\right)^2 + \left(\frac{\partial \varepsilon}{\partial y}\right)^2$. Let's consider the Ansatz:

$$\begin{aligned} X_{21} &= \alpha I \\ X_{12} &= \beta C \\ X_{22} &= \gamma I \end{aligned} \tag{5.18}$$

Substituting in (5.16) and solving for the unknown parameters, we get:

$$\begin{aligned} \alpha &= \sqrt{1-\omega^2} \\ \beta &= -\tfrac{\sqrt{1-\omega^2}}{\omega} \\ \gamma &= \omega \end{aligned} \tag{5.19}$$

Producing the symplectic transformation matrix:

$$\Lambda = \begin{bmatrix} C & -\frac{\sqrt{1-\omega^2}}{\omega} C \\ \sqrt{1-\omega^2} I & \omega I \end{bmatrix} \tag{5.20}$$

Notice that for translations and rotations $\omega^2 = 1$, and in that case the transformation matrix does not change the time variables τ and I.

The conformal transformation matrix C has the property $C^{\dagger}C = \omega^2 I$. In consequence, the transformation matrix Λ is unitary:

$$\Lambda^{\dagger}\Lambda = I \tag{5.21}$$

This means that the transformation is an isometry and it preserves the lengths of all vectors. We would have to select the units of all variables to be conformable. In relativity, the speed of light c makes the time dimension conformable with the space dimensions. Assuming the units of τ and ϕ are units of length, the isometric property gives us for the transformed system:

$$dx'^2 + dy'^2 + d\tau'^2 + dI'^2 = dx^2 + dy^2 + d\tau^2 + dI^2 \tag{5.22}$$

5.6 The dilation transformation

We can now find the matrix corresponding to a dilation transformation $x' = \lambda x,\ y' = \lambda y$:

$$D = \begin{bmatrix} \lambda & 0 & -\sqrt{1-\lambda^2} & 0 \\ 0 & \lambda & 0 & -\sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & 0 & \lambda & 0 \\ 0 & \sqrt{1-\lambda^2} & 0 & \lambda \end{bmatrix} \tag{5.23}$$

We can see that in order to maintain all entries real, the dilation factor is constrained to be $0 < \lambda \leq 1$.

To better understand this transformation let's consider a harmonic oscillator described by action-angle variables (Fetter and Walecka 2003). Calling θ' and Φ' the position and momentum of the oscillator, we have $\psi\left(\theta', \Phi'\right) = \Phi'$ as stream function and as Hamiltonian (5.12):

$$H = \Phi' + \phi' \tag{5.24}$$

Applying the transformation(5.23), we get the Hamiltonian in the new coordinate system:

$$H = \left(\lambda + \sqrt{1-\lambda^2}\right)\Phi + \left(\lambda - \sqrt{1-\lambda^2}\right)\phi \tag{5.25}$$

If we consider the dilation to be small $\lambda = 1 - {}^{\varepsilon^2}\!/_2$ for $\varepsilon << 1$, the Hamiltonian:

$$H = (1+\varepsilon)\,\Phi + (1-\varepsilon)\,\phi + O\left(\varepsilon^2\right) \tag{5.26}$$

The corresponding Hamiltonian equations are:

$$\begin{aligned} \tfrac{d\theta}{dt} &= (1+\varepsilon) \\ \tfrac{d\tau}{dt} &= (1-\varepsilon) \end{aligned} \tag{5.27}$$

With solution $\theta = (1+2\varepsilon)\,\tau + \theta_0 + O\left(\varepsilon^2\right)$. That implies that the time necessary to complete a revolution in the new coordinate system has shrunk by $1-2\varepsilon$.

5.7 Chapter conclusion

In this chapter we looked at canonical transformations of coordinates, those that preserve the form of Hamilton's equations. The Jacobians of these transformations are members of the symplectic group. Then we examined under which circumstances conformal transformations are canonical. We found that the only transformations that are both canonical and conformal are isometries. We then constructed a matrix that is symplectic and includes the Jacobian of a conformal transformation as part of it. We accomplished this by including time as part of the transformed variables and a dummy variable *I* to act as its conjugate variable.

We showed that with the new matrix Λ, we could include dilations as part of the transformations that are both canonical and conformal. This generalizes the linear transformation $w(z) = Az + b$ to be canonical in all cases by producing dilations on the time variable, as hinted in Chapter 4.

Nevertheless, other than for linear transformations, the matrix Λ is not likely to be a Jacobian. In all likelihood, it doesn't exist any other coordinate transformations with the property of being both conformal and canonical.

APPENDIX A

MATHEMATICAL CONCEPTS

A.1 Polar coordinates

The relation between polar and Cartesian coordinates is:

$$\begin{aligned} x &= \rho\,\cos\theta \\ y &= \rho\,\sin\theta \end{aligned} \tag{A.1}$$

The partial differentiation operators are in consequence:

$$\begin{aligned} \tfrac{\partial}{\partial x} &= \cos\theta\tfrac{\partial}{\partial \rho} - \tfrac{\sin\theta}{\rho}\tfrac{\partial}{\partial \theta} \\ \tfrac{\partial}{\partial y} &= \sin\theta\tfrac{\partial}{\partial \rho} + \tfrac{\cos\theta}{\rho}\tfrac{\partial}{\partial \theta} \end{aligned} \tag{A.2}$$

The corresponding inverse transformation:

$$\begin{aligned} \tfrac{\partial}{\partial \rho} &= \cos\theta\tfrac{\partial}{\partial x} + \sin\theta\tfrac{\partial}{\partial y} \\ \tfrac{\partial}{\partial \theta} &= -\rho\sin\theta\tfrac{\partial}{\partial x} + \rho\cos\theta\tfrac{\partial}{\partial y} = -y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y} \end{aligned} \tag{A.3}$$

The 2D Lagrangian operator in polar coordinates:

$$\Delta = \nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2} \tag{A.4}$$

A.2 Bessel functions identities

The Bessel functions of the first kind generating function is:

$$e^{i\gamma\rho\,\sin\theta} = \sum_{n=-\infty}^{\infty} J_n\left(\gamma\rho\right)e^{in\theta} \tag{A.5}$$

From which an integral for the *n*-th term can be found:

$$J_n\left(\gamma\rho\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(\gamma\rho\,\sin\theta-n\theta)}d\theta \tag{A.6}$$

Orthogonality of Bessel functions:

$$\int_0^{\infty}x\,J_m\left(x\right)J_n\left(x\right)dx=\delta_{m,n} \tag{A.7}$$

Parity relations:

$$J_{-n}\left(x\right)=\left(-1\right)^n J_n\left(x\right) \tag{A.8}$$

Recursive relations:

$$\begin{aligned}\tfrac{2n}{x}J_n\left(x\right)&=J_{n-1}\left(x\right)+J_{n+1}\left(x\right)\\ 2\tfrac{d\,J_n(x)}{dx}&=J_{n-1}\left(x\right)-J_{n+1}\left(x\right)\end{aligned} \tag{A.9}$$

Orthogonality in an interval: let $\gamma_{n,j},\ \gamma_{n,k}$ be roots of the *n*-th order Bessel function of the first kind, then:

$$\int_0^1\rho\,J_n\left(\gamma_{n,j}x\right)\ J_n\left(\gamma_{n,k}x\right)dx=\frac{\delta_{i,j}}{2}\left(J_n'\left(\gamma_{n,j}\right)\right)^2 \tag{A.10}$$

Eigenvectors of the Laplace equation over a unit circle, null at the boundary:

$$\nabla^2\psi_{n,k}=-\gamma_{n,k}^2\psi_{n,k} \tag{A.11}$$

$$\psi_{n,k}\left(\rho,\theta\right)=\frac{1}{\sqrt{\pi}J_n'\left(\gamma_{n,k}\right)}J_n\left(\gamma_{n,k}\rho\right)e^{in\theta} \tag{A.12}$$

Eigenvectors form a complete set over the unit circle:

$$\int_{-\pi}^{\pi}\int_0^1\rho\,\psi_{n,k}^*\psi_{m,j}d\rho d\theta=\delta_{n,m}\delta_{k,j} \tag{A.13}$$

Partial derivatives in the *y* axis:

$$\begin{aligned}
&\tfrac{\partial}{\partial y}\psi_{n,k} = \left(\sin\theta\tfrac{\partial}{\partial\rho} + \tfrac{\cos\theta}{\rho}\tfrac{\partial}{\partial\theta}\right)\tfrac{1}{\sqrt{\pi}J_n'\left(\gamma_{n,k}\right)}J_n\left(\gamma_{n,k}\rho\right)e^{in\theta} \\
&= \tfrac{\gamma_{n,k}}{\sqrt{\pi}J_n'\left(\gamma_{n,k}\right)}\left(\tfrac{e^{i\theta}-e^{-i\theta}}{2i}\right)\tfrac{\partial J_n\left(\gamma_{n,k}\rho\right)e^{in\theta}}{\partial\left(\gamma_{n,k}\rho\right)} \\
&+ \tfrac{\gamma_{n,k}\, i}{\sqrt{\pi}J_n'\left(\gamma_{n,k}\right)}\left(\tfrac{e^{i\theta}+e^{-i\theta}}{2}\right)\tfrac{nJ_n\left(\gamma_{n,k}\rho\right)e^{in\theta}}{\gamma_{n,k}\rho} \\
&= \tfrac{\gamma_{n,k}}{\sqrt{\pi}J_n'\left(\gamma_{n,k}\right)}\left[\begin{array}{c}\left(\tfrac{e^{i\theta(n+1)}-e^{i\theta(n-1)}}{2i}\right)\left(\tfrac{J_{n-1}\left(\gamma_{n,k}\rho\right)-J_{n+1}\left(\gamma_{n,k}\rho\right)}{2}\right) \\ +i\left(\tfrac{e^{i\theta(n+1)}+e^{i\theta(n-1)}}{2}\right)\left(\tfrac{J_{n-1}\left(\gamma_{n,k}\rho\right)+J_{n+1}\left(\gamma_{n,k}\rho\right)}{2}\right)\end{array}\right] \\
&= \tfrac{\gamma_{n,k} i}{4\sqrt{\pi}J_n'\left(\gamma_{n,k}\right)}\left[2e^{i\theta(n+1)}J_{n+1}\left(\gamma_{n,k}\rho\right) + 2e^{i\theta(n-1)}J_{n-1}\left(\gamma_{n,k}\rho\right)\right] \\
&= \tfrac{\gamma_{n,k} i}{2\sqrt{\pi}J_n'\left(\gamma_{n,k}\right)}\left(e^{i\theta(n+1)}J_{n+1}\left(\gamma_{n,k}\rho\right) + e^{i\theta(n-1)}J_{n-1}\left(\gamma_{n,k}\rho\right)\right)
\end{aligned} \tag{A.14}$$

$$\frac{\partial^2}{\partial y^2}\psi_{n,k} = -\frac{\gamma_{n,k}^2}{4\sqrt{\pi}J_n'\left(\gamma_{n,k}\right)}\left(e^{i\theta(n+2)}J_{n+2}\left(\gamma_{n,k}\rho\right) + 2e^{in\theta}J_n\left(\gamma_{n,k}\rho\right) + e^{i\theta(n-2)}J_{n-2}\left(\gamma_{n,k}\rho\right)\right) \tag{A.15}$$

A.3 Complex plane identities

Following Bazant (2004), we define the gradient in terms of complex numbers as:

$$\nabla f = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \tag{A.16}$$

And its conjugate:

$$\bar{\nabla} f = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} \tag{A.17}$$

Now applying this operators to an analytic function $f\left(x, y\right) = g\left(x, y\right) + i\, h\left(x, y\right)$:

$$\nabla f = \left(\frac{\partial g}{\partial x} + i\frac{\partial h}{\partial x}\right) + i\left(\frac{\partial g}{\partial y} + i\frac{\partial h}{\partial y}\right) = \left(\frac{\partial g}{\partial x} - \frac{\partial h}{\partial y}\right) + i\left(\frac{\partial h}{\partial x} + \frac{\partial g}{\partial y}\right) = 0 \tag{A.18}$$

where the last identity follows from the Cauchy-Riemann conditions. Similarly:

$$\begin{aligned}&\bar{\nabla} f = \left(\frac{\partial g}{\partial x} + i\frac{\partial h}{\partial x}\right) - i\left(\frac{\partial g}{\partial y} + i\frac{\partial h}{\partial y}\right) = \left(\frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}\right) + i\left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right)\\ &= 2\frac{\partial g}{\partial x} + 2i\frac{\partial h}{\partial x} = 2f'(z)\end{aligned} \tag{A.19}$$

The Laplacian operator is easily built:

$$\begin{aligned}&\nabla^2 f = \bar{\nabla}\nabla f = \nabla\bar{\nabla} f = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f =\\ &\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\end{aligned} \tag{A.20}$$

Identity (A.18) makes it easy to see that for analytic functions $\nabla^2 f = 0$.

Sometimes instead of working in the *x,y* coordinates it is more convenient to work on terms of the coordinates:

$$\begin{aligned}z &= x + iy\\ \bar{z} &= x - iy\end{aligned} \tag{A.21}$$

Applying the chain rule we find:

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\\ \frac{\partial}{\partial y} &= i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}\end{aligned} \tag{A.22}$$

And its inverse:

$$\begin{aligned}\frac{\partial}{\partial z} &= \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}\\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}\end{aligned} \tag{A.23}$$

From it, it immediately follows an expression of the Laplacian in terms of the complex coordinates (A.21):

$$\nabla^2 f = 4\frac{\partial^2 f}{\partial z\,\partial \bar{z}} \tag{A.24}$$

The gradient operators in complex coordinates:

$$\begin{aligned}\nabla f &= \tfrac{\partial f}{\partial x} + i\tfrac{\partial f}{\partial y} = \left(\tfrac{\partial f}{\partial z} + \tfrac{\partial f}{\partial \bar{z}}\right) + i\left(i\tfrac{\partial f}{\partial z} - i\tfrac{\partial f}{\partial \bar{z}}\right) = 2\tfrac{\partial f}{\partial \bar{z}} \\ \bar{\nabla} f &= \tfrac{\partial f}{\partial x} - i\tfrac{\partial f}{\partial y} = \left(\tfrac{\partial f}{\partial z} + \tfrac{\partial f}{\partial \bar{z}}\right) - i\left(i\tfrac{\partial f}{\partial z} - i\tfrac{\partial f}{\partial \bar{z}}\right) = 2\tfrac{\partial f}{\partial z}\end{aligned} \tag{A.25}$$

An interesting case is the Laplacian applied to a radial function:

$$\begin{aligned}&\nabla^2 f\left(r^2\right) = 4\tfrac{\partial^2 f(z\,\bar{z})}{\partial z \partial \bar{z}} = 4\tfrac{\partial}{\partial z}\left(z\,f'\left(z\,\bar{z}\right)\right) = 4f'\left(z\,\bar{z}\right) + 4z\,\bar{z}\,f''\left(z\,\bar{z}\right) = \\ &4f'\left(r^2\right) + 4r^2 f''\left(r^2\right)\end{aligned} \tag{A.26}$$

A similar result is obtained using the Laplacian in two-dimensional radial coordinates.

$$\nabla^2 f\left(r^2\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) f\left(r^2\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(2r^2 f'\left(r^2\right)\right) = 4f'\left(r^2\right) + 4r^2 f''\left(r^2\right) \tag{A.27}$$

Another interesting relation is to apply the conjugate gradient operator (A.17) to a non-analytic function $F = f\,(x, y) + i\,g\,(x, y)$.

$$\bar{\nabla} F = \frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y} = \left(\frac{\partial f}{\partial x} + i\frac{\partial g}{\partial x}\right) - i\left(\frac{\partial f}{\partial y} + i\frac{\partial g}{\partial y}\right) = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) + i\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \tag{A.28}$$

We can identify in the right side the divergence and curl of F, allowing us to write the expression:

$$\bar{\nabla} F = 2\frac{\partial F}{\partial z} = div\,F + i\,curl\,F \tag{A.29}$$

Now consider expressing the advection operator:

$$\vec{u}\cdot\nabla c = \frac{\partial \psi}{\partial y}\frac{\partial c}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial c}{\partial y} \tag{A.30}$$

We express it in terms of the complex gradient. Let's start with the scalar product of the complex gradients:

$$\bar{\nabla}\psi\,\nabla c = \left(\frac{\partial\psi}{\partial x} - i\frac{\partial\psi}{\partial y}\right)\left(\frac{\partial c}{\partial x} + i\frac{\partial c}{\partial y}\right) = \frac{\partial\psi}{\partial x}\frac{\partial c}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial c}{\partial y} + i\left(\frac{\partial\psi}{\partial x}\frac{\partial c}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial c}{\partial x}\right) \tag{A.31}$$

From which we can see that the advection operator is:

$$\vec{u}\cdot\nabla c = Im\left[\bar{\nabla}\psi\,\nabla c\right] = 4Im\left[\frac{\partial\psi}{\partial z}\frac{\partial c}{\partial\bar{z}}\right] \tag{A.32}$$

A.4 Conformal invariance of the advection operator

Consider the advection operator applied to a scalar field c, where the velocity field $\vec{u}$ is generated by the stream function ψ:

$$\vec{u}\cdot\nabla c = \frac{\partial\psi}{\partial y}\frac{\partial c}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial c}{\partial y} = \begin{bmatrix}\frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial c}{\partial x}\\ \frac{\partial c}{\partial y}\end{bmatrix} \tag{A.33}$$

Applying the conformal transformation $x(\varepsilon,\eta)\,,\,y(\varepsilon,\eta)$ by means of the chain rule:

$$\begin{aligned}\vec{u}\cdot\nabla c = \begin{bmatrix}\frac{\partial\psi}{\partial\varepsilon} & \frac{\partial\psi}{\partial\eta}\end{bmatrix}\begin{bmatrix}\frac{\partial\varepsilon}{\partial x} & \frac{\partial\varepsilon}{\partial y}\\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial\varepsilon}{\partial x} & \frac{\partial\eta}{\partial x}\\ \frac{\partial\varepsilon}{\partial y} & \frac{\partial\eta}{\partial y}\end{bmatrix}\begin{bmatrix}\frac{\partial c}{\partial\varepsilon}\\ \frac{\partial c}{\partial\eta}\end{bmatrix} = \\ \begin{bmatrix}\frac{\partial\psi}{\partial\varepsilon} & \frac{\partial\psi}{\partial\eta}\end{bmatrix}\begin{bmatrix}0 & -\frac{\partial\varepsilon}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\varepsilon}{\partial y}\frac{\partial\eta}{\partial x}\\ \frac{\partial\varepsilon}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\varepsilon}{\partial y}\frac{\partial\eta}{\partial x} & 0\end{bmatrix}\begin{bmatrix}\frac{\partial c}{\partial\varepsilon}\\ \frac{\partial c}{\partial\eta}\end{bmatrix}\end{aligned} \tag{A.34}$$

And from (A.32):

$$\vec{u}\cdot\nabla c = 4Im\left[\frac{\partial\psi}{\partial z}\frac{\partial c}{\partial\bar{z}}\right] = 4\left|\frac{dw}{dz}\right|^2 Im\left[\frac{\partial\psi}{\partial w}\frac{\partial c}{\partial\bar{w}}\right] \tag{A.35}$$

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