Restructuring for Mathematical Power: Techniques for Teaching Thinking in Algebra

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RESTRUCTURING FOR MATHEMATICAL POWER:
TECHNIQUES FOR TEACHING THINKING IN ALGEBRA

A Thesis Presented
by
BARBARA D. NELSON

Submitted to the Office of Graduate Studies and Research of
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TECHNIQUES FOR TEACHING THINKING IN ALGEBRA

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This thesis is dedicated to my husband, Garth Nelson, who believed it could be done and who did so much to smooth the way. My success is rooted in his support.
ABSTRACT

RESTRUCTURING FOR MATHEMATICAL POWER:
TECHNIQUES FOR TEACHING THINKING IN ALGEBRA

DECEMBER, 1992

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Recent critiques of mathematics education have resulted in proposals to restructure learning and teaching for mathematical power. The new vision pictures the classroom as a community of learners where mathematics comes alive as a useful tool in our technological society. However, many high school mathematics teachers are struggling to understand and implement the fundamental instructional change inherent in the vision. Written from the perspective of a high school teacher for experienced high school mathematics teachers, this thesis attempts to bridge vision and practice.

To clarify the vision, current literature on reform in mathematics education is synthesized into a framework of eight instructional targets. Four of the targets focus on student behavior indicative of mathematical power: deep understanding of concepts and schemas, mathematical thinking, communication about mathematics and a positive disposition toward mathematics. The other four targets focus on the instructional setting: student-centered tasks,
a variety of work formats, mathematical tools and assessment alternatives. Suggestions for each target help teachers generate ideas for implementation.

The framework is based on seven learning principles synthesized from current research: 1) knowledge is constructed; 2) all students can grapple with complex ideas; 3) conceptual learning is effective; 4) prior knowledge influences learning; 5) learning is a social act; 6) change in cognitive structure is a goal of teaching and 7) students must be actively engaged to learn.

To implement the vision, the recommended strategy for experienced teachers is to expand their repertoire of instructional methods by focusing on teaching thinking. Guidelines for a model of thinking, levels of curriculum planning and relevant issues in cognitive education are incorporated into a lesson plan model.

As tactical examples of the implementation strategy, three techniques designed to develop the thinking processes of classifying, pattern finding and concept formation are modeled using Algebra I content. The presentation of the techniques is structured to emphasize general instructional decisions made by the teacher in order to enhance transfer to particular classrooms.

Two underlying convictions are: experienced teachers attempting reform must focus on the process of instruction; and successful reform depends on teacher reflection leading to ownership of the vision.
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Reflection

Mathematical Thinking

Rationale

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Brainstorm

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Reflection

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Rationale

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CHAPTER I
THE CHALLENGE AND THE STRUGGLE

Overview

This thesis addresses the challenge faced by classroom teachers to implement a new vision of mathematics education. It is written for experienced high school mathematics teachers who are struggling to respond to calls for reform from various professional groups. It reviews the challenge, but focuses on restructuring the processes of doing and learning mathematics to help students achieve the goal of mathematical power. A framework of instructional targets is synthesized as an interpretation of the new vision. As a strategy for remodeling instruction, a focus on teaching thinking is proposed and, as tactics for implementation, instructional techniques are elaborated in a manner to ease transfer.

This introductory chapter begins by reviewing calls for change in education in general and in mathematics education in particular. The vision of educating for mathematical power that has emerged in recent years is described. Mathematical power is defined. Proposed changes in curriculum and instruction are outlined and the accompanying sense of renewal and excitement is shared.

Despite the climate of change, high school mathematics teachers face many difficulties when attempting
to take the vision into the classroom. Barriers include traditional teaching models, the difficulty of altering an individual’s teaching style and the method of presenting examples of reform.

In conclusion, a two part approach for teachers attempting restructuring is recommended. Teachers need a clear understanding of the vision, so research is synthesized in a framework of instructional targets. Teachers need a strategy and tactics for implementation, so a focus on the teaching of thinking is proposed and transferable techniques are designed to add to a teacher’s instructional repertoire. For the high school teacher this is a realistic response to the calls for change.

**Calls for Change**

**Education in General.**

Change is a constant in effective education. As society evolves, education responds to new needs and goals. Recent calls for change in American education reflect the challenge of preparing students for the twenty-first century. During the next few decades the global community will continue to shift from an industrial to an informational society with a rapidly changing knowledge base. Workers will hold a series of jobs which are increasingly dependent on technology. Employers will reward high performance work dependent on thinking skills,
interpersonal skills, application of technology, productive allocation of resources and management of information (Packer 1992). Informed citizenship will require reading, interpreting and evaluating technological information on complex issues. Citizens will need to think effectively to shape the future. Yet, recent national reports have described the United States "as a 'nation at risk' because we [Americans] are failing to provide students with the most basic component of education--instruction that fosters the ability to think" (Halpern 1989, 3).

To prepare today's students for tomorrow, educational futurists suggest new goals and strategies for curriculum and instruction. Recurring themes include active learning, higher cognitive skills, lifelong learning, holistic education, diversification of students, education across the disciplines, a shift from content to process and communication skills (Benjamin 1989). Reformers have responded with proposals for restructured schools, national standards, school choice, cooperative learning, learning styles instruction, mastery learning and the implementation of total quality principles. Such "bold attempts to rethink our schools, the ways that we teach and the ways students learn...are both frightening and exhilarating" (Glickman 1992, 1).
Critiques of mathematics education epitomize the challenges to education. As prospective citizens and workers in the twenty-first century, students need preparation in the mathematical sciences. Technology has transformed the workplace; the use of calculators and computers is commonplace. Industry needs employees who are confident in their ability to use mathematics to formulate and explore problems. Statistics influence decisions on public policy. To function as an informed citizen, numeracy (mathematical literacy) is as significant as verbal literacy. "The information age is a mathematical age" (National Research Council 1989, 74).

The discipline of mathematics will continue to change as society enters the twenty-first century. "During the past fifty years, more mathematics has been created than in all previous ages put together" (Stewart 1987, 13). Applications of this new knowledge permeate the social and life sciences. School mathematics must extend beyond the algebra-geometry-precalculus-calculus sequence which feeds the physical sciences and engineering. Mathematics with an emphasis on theoretical abstraction and the physical sciences has evolved into the mathematical sciences with a multiplicity of applications.

National evaluations and international comparisons report that the mathematics education of American students
is not keeping pace. Three of every four American students never acquire the mathematics needed as prerequisites for jobs or college (National Research Council 1989). American students "rank at the bottom on most international tests—behind children in Europe and East Asia" (Magaziner and Clinton 1992, 10). The mathematics achievement of the top five percent of American students is equaled by the top fifty percent of Japanese students (National Research Council 1989). A perception of deficiency grows as terms such as innumeracy (Paulos 1988) and math anxiety (Tobias 1980) become part of the language. These concerns are echoed by professionals in education.

Responding to the Calls for Change.

Mathematics educators are responding to these analyses through national commissions, professional organizations and state departments of education. In Everybody Counts: A Report to the Nation on the Future of Mathematics Education, the National Research Council (1989) delineates the crisis in mathematics education and outlines a broad strategy. It calls for new curriculum standards, for upgraded teaching and for responsive assessment approaches. Continuing this work, the Mathematical Sciences Education Board and the National Research Council (1990) provide a rationale for "a new practical philosophy of mathematics" (iii) in Reshaping School Mathematics: A Philosophy and Framework for Curriculum.
The National Council of Teachers of Mathematics (NCTM) is taking the lead in establishing broad goals for curriculum, teaching and evaluation. This organization has produced two documents, *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Professional Standards for Teaching Mathematics* (1991), which are benchmarks for providing objectives that build on present knowledge and practice. The National Council of Supervisors of Mathematics (1988) endorses the NCTM Standards by redefining its position on the essential components of mathematics education. Emphasized in both of these endeavors are problem solving, communication, mathematical reasoning and the application of mathematics to everyday situations. A non-threatening learning climate and the evaluation of problem solving and reasoning are steps to these competencies.

The consensus at the national level is intentionally broad in scope as the success of the vision depends on local implementation. *Mathematics Framework for California Public Schools: Kindergarten through Grade Twelve* (California State Department of Education 1991) reinforces and translates the call for change into practices to be adopted at the state level. Ohio has developed a model curriculum based on the NCTM Standards, as well.

These proposals focus on the restructuring of learning and teaching. They are "nothing less than a call to revolution--a call that is being heard and heeded"
(Willis 1992). But what is this ideal image for which the revolt is staged?

The New Vision

Mathematical Power.

Defining the goal. Mathematical power is the force that drives the new vision of mathematics education. Mathematical power is "an individual's abilities to explore, conjecture and reason logically as well as the ability to use a variety of mathematical methods effectively to solve non-routine problems" (NCTM 1989, 5). Mathematical power includes the self confidence and disposition to exercise these abilities.

Mathematical power encompasses knowledge, skill and affect. Knowledge of mathematical concepts and properties is only one element. Skills in higher-order thinking, in communicating processes and results and in using tools and techniques such as calculators, computers, manipulatives and procedural algorithms are other elements. Students work individually or in groups with confidence and enthusiasm. There is appreciation of the historical and social role of the discipline. "Mathematically powerful students think and communicate, drawing on mathematical ideas and using mathematical tools and techniques" (California State Department of Education 1991, 2).
The classroom. The vision pictures the mathematics classroom as a place where mathematics comes alive as a useful tool in our technological society. Various formats actively engage students in doing mathematics. Desks are clumped together as most students work in small groups. Students explore with concrete models as they try to identify patterns. Or student pairs cluster at computers using a spreadsheet to determine an optimal combination for a formula with several variables. Another day the room may resemble an art class as students create designs based on linear equations. Or a debate may develop on how to analyze and present data collected in a student designed survey. Students are exploring, creating, thinking and problem solving using mathematics with confidence.

The classroom is a mathematics community. In a risk free environment, mathematics is studied as "a science and language of patterns. To know mathematics is to investigate and express relationships among patterns" (Mathematical Sciences Education Board and National Research Council 1990, 16). The aim of mathematics education is to make sense of the patterns in the real world. Goals are different for different levels--for the elementary level number sense, for the secondary level symbol sense and for higher education function sense (National Research Council 1989). However, the purposeful use of knowledge and skills is central for all.
The curriculum. The vision shifts the curricular emphasis from content to mathematical processes. "Knowing mathematics is doing mathematics" (NCTM 1989, 7). Computation is dethroned by mathematical thinking. The processes of reasoning, problem solving, communicating and making connections are built into curricula. Discussions focus on paths to solutions more than on final results. Skills and techniques are introduced as tools in a problem-rich curriculum.

Content becomes the context for learning mathematical processes. The curriculum reflects the nature and role of the mathematical sciences as they evolve into the twenty-first century. At the secondary level, statistics, probability and discrete mathematics are prioritized with algebra and geometry. To explore connections, the use of mathematics across the curriculum is promoted.

New Roles.

The vision of mathematics education casts teachers as instructional decision makers and learning coaches. Students are active learners taking responsibility for their own education. Teachers guide, clarify and question; students investigate, construct and represent. They form a partnership in developing the students' mathematical power.

All students share the promise of the vision. Mathematics is perceived as valuable to the future success of many. Computational proficiency is not a prerequisite
to other areas of study. Though the depth and speed of coverage may vary, students study the same basic topics. All students are capable of and have a right to education for mathematical literacy.

**Challenge and Renewal.**

As this vision of mathematics education spreads, a climate of challenge and renewal emerges. A sense of purpose and opportunity pervades the professional literature and conferences.

No longer can we afford to sit idly by while our children move through school without receiving mathematical preparation appropriate for the twenty-first century. The challenges are clear. The choices are before us. It is time to act. (National Research Council 1989, 96)

Yet, as many march with the Standards to a new sense of fulfillment, other high school mathematics teachers feel left behind. These teachers are struggling to understand and implement the vision.

**Problems of Implementation**

**Understanding the Vision.**

The quest begins as high school teachers become learners who are trying to interpret, understand and synthesize the many representations of the new vision of mathematics education. Time must be found to read, reflect
and exchange ideas about a large and growing body of knowledge.

As knowledge deepens, the awareness grows that the vision requires much more than adding a unit on probability to the algebra curriculum or substituting a course in discrete mathematics for one in precalculus. The vision requires a model of instruction different from the model experienced teachers have known as professionals. Furthermore, "the kind of teaching envisioned...is significantly different from what many teachers themselves have experienced as students in mathematics classes" (NCTM 1991, 2).

The traditional model of instruction. Historically, mathematics teaching has been guided by the nutritionist model.

Teachers are seen as technical experts who impart privileged knowledge to students....Children are fed portions of knowledge, in measured doses. They are expected to digest it and to give evidence, in class response and examination, that they have done so. (Schon 1983, 327)

Mathematics teachers are transmitters of inert knowledge. The traditional model emphasizes paper and pencil calculation and symbol manipulation with the goal of preparation for future mathematics courses. Mastery of specified procedures precedes contrived applications. Mathematics is fragmented into isolated fields. Teachers teach what is presented in the textbook and students learn what they think will be on the test. The teacher functions
in an authoritarian mode like "Moses coming down from Mt. Sinai" (National Research Council 1989, 66).

In a 1979 report on a series of National Science Foundation studies on mathematics and science education, the following remarks were cited as typical of almost all classrooms observed.

In all math classes I visited, the sequence of activities was the same. First, answers were given for the previous day's assignment. The more difficult problems were worked by the teacher or a student at the chalkboard. A brief explanation, sometimes none at all, was given of the new material, and problems were assigned for the next day. The remainder of the class was devoted to working on the homework while the teacher moved about the room answering questions. The most noticeable thing about math classes was the repetition of this routine. (Fey 1979, 494-495)

This pattern is still prevalent in the 1990's (NCTM 1991; Driscoll and Lord 1990). The traditional model does not fit the proposals for mathematics education.

Contrast with the vision. With this background, many experienced secondary mathematics teachers have had no contact with the model of instruction painted in the vision. "Metaphorically speaking, the mathematics teacher ought to be less of a nutritionist in instruction, and more of a guide, coach, and psychologist" (Driscoll and Lord 1990, 239). Teachers need experience with the presentation of math concepts in the context of problems, with student-centered activities, with question strategies that elicit higher-order thinking and with instruction that integrates calculators and computers.
A new mindscape. The vision asks teachers to restructure the model by which they have taught and been taught. A new mindscape is required. A mindscape is a paradigm through which one sees the world and one's place in it. Mindscapes are mainly implicit and have an enormous bearing on behavior. "In a very special way mindscapes are intellectual security blankets on the one hand and road maps through an uncertain world on the other" (Sergiovanni 1985, 5). The realization that the vision requires a fundamental change in one’s behavior can be overwhelming.

Other obstacles. Though reworking the patterns by which one functions is a major task, the traditional model presents classroom teachers with other obstacles.

The concept of privileged knowledge which it is the business of teachers to teach, and students to learn...is embodied in text, curriculum, lesson plans, examinations; indeed it is institutionalized in every aspect of the school. (Schon 1983, 329) Administrators, parents, publishers and test makers still function with the traditional paradigm. Many parents take the view that what was good for them is good for their children. Textbook publishers often pay lip service to the new trends without real change. National standardized tests focus on arithmetic skills, algebra and geometry. The classroom teacher may be attempting change with minimal support and inappropriate materials, while being evaluated
by administrators and skeptical parents on the basis of standardized test scores.

Lack of time and isolation are other obstacles. "Time is the enemy....just everyday preparation and paperwork take an enormous amount of time....It takes a great deal of extra time and energy to try something new in the classroom." (Henderson 1987, 153). Professionally most high school teachers are isolated. Yet, staff collegiality has been identified as important to the success of school mathematics programs (Driscoll and Lord 1990).

Despite the hurdles, many mathematics teachers still want to be agents of change. They want to improve their teaching. They want their students to feel successful and to appreciate the richness of mathematics. They believe the vision is a worthwhile, if not necessary, goal. "The prospects are frightening and exhilarating. But at last citizens and school people are willing to do what we have not done easily before: take risks" (Glickman 1992, 1).

Equipped with understanding of and commitment to the vision, how do these change agents make the vision a classroom reality?

Taking the Vision into the Classroom.

A strategy. As few secondary teachers have the authority or resources to implement the vision, a coordinated effort involving teachers, parents, administrators, public officials, university faculty and
business leaders is advocated (National Research Council 1989). Yet, many high school mathematics teachers are in situations that do not fit this ideal. For them, a realistic strategy is to focus on one aspect of the vision, then attempt implementation with the resources available.

However, focusing on one aspect of the vision may be difficult when reformers call for change in many areas. Teachers are encouraged to use cooperative learning, to have students keep math journals, to furnish problems rich in appropriate mathematics, to assess mathematical thinking, to provide projects for different learning styles and much more. Teachers reluctant to start with only one aspect of the vision may be heartened by the realization that tackling one aspect usually incorporates change in other areas.

**Tactics.** Once a focus for implementation is identified, models can be consulted for tactics.

Exemplary curriculum materials can help teachers think about their current roles, try out new roles, and modify the way they teach by drawing directly on the accumulated experience of teachers who have helped to develop and try out these materials. (Lovitt et al. 1990, 230)

However, model lessons usually are written for a particular instructional situation. The presentation of an example may ignore its application to a different level or course. The honors geometry teacher may dismiss the lesson set in a standard general mathematics class. Or an example based on the assumption that students have prior experience
with cooperative learning may fail when students are unfamiliar with this format. When models are presented as situation specific examples, the teacher faces an impediment to transfer.

A Response to the Needs of Classroom Teachers

This thesis is an attempt to bridge theory and practice in mathematics education. It takes the view that success in achieving mathematics reform depends on meeting the needs of the classroom teacher. This thesis is written from the perspective of an experienced high school teacher for other experienced high school mathematics teachers who are struggling to be agents of change given the challenges outlined in this chapter.

In the body of this thesis, the implementation of the new vision of mathematics education is tackled in two stages. First, teachers must understand the vision. In Chapter II, current literature on mathematics education is synthesized as a framework of instructional targets. Second, teachers must develop a strategy and tactics to expand their repertoire of instructional techniques. In Chapter III, a focus on the teaching of thinking is proposed as the strategy by which experienced secondary mathematics teachers can remodel their mindscapes of instruction. In Chapter IV, techniques are presented as tactical models of this strategy.
Chapter V concludes with a discussion of two convictions: 1) restructuring should focus on student process skills and on expansion of the traditional processes of instruction and 2) teacher reflection and ownership are necessary to successful implementation of the new vision. These beliefs underlie the suggestions made throughout the thesis.
Overview

Professional organizations, governmental boards, educational experts and curriculum developers outline the new vision of mathematics education from broad goal statements to specific applications. Furthermore, an extensive body of research and its analysis lies behind the recommendations. Consequently, pursuing the vision entails interpreting, evaluating and applying these suggestions. This is a potentially overwhelming task for any teacher.

This chapter provides a synthesis of the mathematics education literature into a framework of broad goal categories referred to throughout this thesis as `targets'. The framework of eight targets serves as a device to organize information for teachers struggling to clarify the vision.

Two background sections precede the explanation of the targets. The first section outlines implications of choosing a framework format and instructional targets. The second summarizes, as seven principles, research about teaching and learning which emerged during the 1980's. Included are: construction of knowledge, complexity for all students, effectiveness of conceptual knowledge, prior knowledge, social aspects of learning, change in cognitive
structure and active nature of learning. These learning principles are the foundation of the framework of targets.

The vision is clarified through the framework of instructional targets. The following eight targets are discussed:

1.) deep understanding of concepts and schemas,
2.) mathematical thinking,
3.) communication about mathematics,
4.) positive disposition toward mathematics,
5.) student-centered tasks,
6.) variety of work formats,
7.) mathematical tools and
8.) assessment alternatives.

Separate sections for each begin with a description of the instructional target. A rationale links the target to current learning research and the needs of the twenty-first century. ‘Options’ or suggestions for each target are given to help teachers generate creative ideas for target implementation.

The two concluding sections discuss student and teacher roles and traditional content in relation to the vision as represented by the framework. The impact of the targets on teacher and student roles is analyzed. The emphasis on restructuring instruction rather than content is defended.
Assumptions behind the Framework

Clarifying the Vision.

Reform documents are no guarantee of change in mathematics education. Essential to success is the practitioner's clarity of the vision of reform.

Any program that seeks to enhance the quality of teaching and learning in mathematics must allow teachers to develop, in practical terms, a clear vision of what these changes mean for their classroom practice and professional growth. (Lovitt et al. 1990, 231)

The broad directions of the reform documents and plethora of suggestions are synthesized here in a structure of broad goal categories called 'targets'. The targets are intended to focus components of instruction: tasks, discourse, climate and analysis (NCTM 1991).

The Structure: Framework, Targets, Options.

A framework. Assumptions are made in interpreting the vision of mathematics education as a framework. First, a framework is a structure outlining the general shape of the vision. The targets are goal categories, not detailed instructions. Thus, use of the framework requires elaboration. Differences in individuals, variations in situations and evolution over time result in diversity. Also, the concept of a framework allows teachers to embellish and tailor the targets to a particular learning situation.
An instructional framework. The framework structures instructional targets, not objectives for student behavior. The teacher attempting restructuring will find it helpful to focus on the conditions of instruction rather than student behavior. Hence, this instructional emphasis makes an important, albeit subtle, distinction. From the perspective of a teacher struggling to implement the vision, clarity and control are implied in the phrase 'provide opportunities for students to' rather than 'the student will'. The implied control comes as a relief in the early stages of acting as a change agent.

This distinction parallels the NCTM's decision to publish two sets of standards. "The Curriculum and Evaluation Standards for School Mathematics represents NCTM's vision of what students should learn in mathematics classrooms" (NCTM 1991, 19). The Professional Standards for Teaching Mathematics presents "a vision of what teaching should entail to support the changes in curriculum set out in the Curriculum and Evaluation Standards" (NCTM 1991, vii).

Targets and options. The selection of the terms 'target' and 'option' for this thesis reflect assumptions about the intended use of the framework. Targets are defined as broad goal categories. This label reinforces the idea that targets are classifications indicating a
general direction. Each target can be attempted in a variety of ways.

Options are ideas, suggestions or prescriptions for implementing targets. The suggestions made are only a sampling of the many options for implementation. The options are not inclusive or prioritized. They are instructional tactics judged as worthwhile and realistic starting points for implementing the vision in the high school mathematics classroom. Furthermore, to enrich the understanding of the vision with examples, the options help meet the challenge of restructuring by stimulating reflection and creativity.

Emerging Views of Teaching and Learning

Research in the 1980’s.

Research which emerged during the 1980’s shaped the restructuring of mathematics teaching and learning. Some of the research was new. Some existed for decades, but only recently received attention. Contributions came from psychologists, cognitive psychologists and educational researchers. When findings indicated learning as domain specific (Anderson 1990), specialists in the psychology of learning mathematics appeared.
Foundations of the Targets.

The vision requires expansion of teaching and learning beyond presentation and memorization of static knowledge. The learning and pedagogical principles upon which the targets are built reflect this focus on concepts and schemas. A concept is an idea abstracted from "experiences which have something in common" (Skemp 1987, 11). Concepts imply understanding beyond rote definition. Schemas (Skemp 1987, Anderson 1990) or frames (Davis 1984) are conceptual structures. Schemas enrich concepts by refining meaning, revealing complexity and delineating links with other concepts.

For the purposes of this thesis, the seven learning principles presented below embody selected research from the 1980's. Each principle cuts across targets and several principles are logical consequences of others. The principles represent general conclusions from constructivist research on learning and teaching. These general pedagogical principles are assumed valid for the mathematics classroom. This perspective forms the foundation of the framework of targets.

Construction of knowledge. Students construct their knowledge of concepts and schemas. "Human beings are theory builders; from the beginning we construct explanatory structures that help us find the deeper reality underlying surface chaos" (Carey 1985, 194). Learners make
sense of the world by incorporating or 'building' new information into existing schemas and then testing the new constructions. The building phase progresses through a sequence of concept representations, namely, concrete, pictorial and abstract (Resnick and Ford 1981). The testing stage of "constructive learning involves 'trying out' ideas, testing to see which solutions work and which do not" (Resnick and Ford 1981, 191). Research indicates that reflective thinking is the means by which concepts and schemas are created and tested. "Reflection is the bootstrap for the construction of mathematical ideas" (Confrey 1990, 116).

Construction of knowledge involves an interplay between existing schemas and new information. If the new information does not mesh with the existing schemas, a state of disequilibrium results. When the latter happens, learners reorganize their knowledge structure to accommodate the new information. "The reorganization of knowledge results in a new way of thinking and understanding that is accompanied by inner satisfaction" (Labinowicz 1985, 18).

Complexity for all students. Every student at all grade levels is capable of dealing with higher level concepts and schemas. Until recently, mastery of computation and symbol manipulation were thought to precede instruction about complex, abstract concepts and schemas.
However, cross cultural research and studies on learning support early presentation of higher-order mathematics. There is abundant evidence that mastery of necessary skills is rarely sufficient for solving complex problems. Moreover, many other countries introduce students to complex problems well before they have studied all the prerequisite skills. Those students often invent effective approaches to the problem, thereby gaining valuable experience in higher-order thinking. (National Research Council 1989, 60)

Furthermore, in speaking about his work with children as both a psychologist and a mathematician, Skemp states that "observations have led me to view with admiration the level of thinking of which children are capable, if we allow them to preserve their natural abilities" (1987, 140).

Effectiveness of conceptual knowledge. Conceptual knowledge is more effective in low level and high level learning than procedural knowledge. Conceptual knowledge refers to concepts and schemas based on relational, rather than instrumental, understanding.

By the former (relational understanding) is meant what I have always meant by understanding, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons,' without realizing that for many pupils and their teachers the possession of such a rule, and ability to use it, was what they meant by 'understanding'. (Skemp 1987, 133)

Schemas aid retention (Anderson 1990) and have superior transfer when compared with rote learning (Meyer 1982). Conceptual knowledge is the path to mastery as
schemas are able to incorporate quantities of new information.

In mathematics too, some of the connections to be formed are associative, for example, the connection between a number concept and its symbol. But the great majority of the connections are conceptual. If, as happens all too often, associative (rote) learning is used, there is a great loss of efficiency and increase of labour involved. (Skemp 1987, 120-121)

A focus on meaning and understanding must replace mere rote learning of computation, symbol manipulation and paper and pencil drill.

Conceptual knowledge and deep understanding promote both low level and high level mathematics learning. Studies show that competence in low level procedural skills is more quickly reached when preceded by instruction in conceptual knowledge. In addition, meaning and understanding are key elements in developing higher-order learning (Peterson 1988).

Prior knowledge. Learning is influenced by prior knowledge. Existing schemas are used to fill-in, and even distort, material during learning (Anderson 1990).

What people learn is never a direct replica of what they have read or been told or even of what they have been drilled on. We know that to understand something is to interpret it and further that an interpretation is based partly on what we’ve been told or have read but also on what we already know.... (Brandt 1988/1989, 15)

Key in effective instruction is the assessment of a student’s prior knowledge and the building of instruction from the student’s reality.
Teachers need to be aware of what students already know and to be alert for misconceptions that require correction. Inappropriate or non-existing schemas are viewed as the bases of systematic student errors or bugs (Davis 1984). Also, incorporation of applications that touch existing schemas activates the motivational aspect of prior knowledge.

Social aspects of learning. Working with others benefits learning. Social groups provide motivation, support, modeling and coaching (Nolan and Francis 1992). Recent research also indicates that communication is necessary in constructing knowledge, as conceptual knowledge is tested through comparison with other people (Skemp 1987). The benefit is greater when students test schemas with other students, rather than with teachers.

In this process, Piaget suggests, the disagreement of adults is less influential than the disagreement of children who are close to them in age and general conceptual level. If this is the case, then children’s learning depends to an important degree on the social environment and the opportunity it provides to interact with peers over intellectual tasks. (Resnick and Ford 1981, 191-192)

Educational research indicates that students learn more in cooperative learning groups with individual accountability and group interdependence (Slavin 1989/1990). When compared with control groups, students taught in cooperative learning formats attained higher achievement on standardized tests, developed a more
positive attitude toward mathematics and gained more in self-confidence (Slavin 1990).

Similar studies isolating gifted students are not conclusive. However, gifted students, learning in heterogeneous cooperative groups, demonstrate no lower achievement than "bright students working alone, competitively or individualistically" (Johnson and Johnson 1987, 169). The evidence that gifted students achieve more is not conclusive.

Change in cognitive structure. The role of the teacher is to stimulate change in the student's cognitive structure (Nolan and Francis 1992). Teaching means providing activities that change a student's concepts and schemas as well as a student's behavior.

Changes in observable behavior are important because they can be used to infer that the learner's cognitive structure has changed, but changes in behavior are an indicator of learning and a result of learning, not the learning itself. (Nolan and Francis 1992, 47)

This is the basis for indirect teaching which prompts and guides intelligent learning (Skemp 1987). Piaget suggests that clinical interaction is the ideal model for teaching. However, "it requires the kind of solid understanding of the subject matter that allows the teacher to recognize sensible but unusual responses and to invent problems that probe a child's understanding" (Resnick and Ford 1981, 193).
Active nature of learning. The student must be an active learner. An innate tendency toward seeking structure as postulated by Gestalt theorists supports a natural drive toward concept building (Resnick and Ford 1981). However, active participation by students must be supported. The teacher cannot do the work of learning for the student. As the focus shifts to skills necessary to form concepts and schemas, what is required is "a participatory link between self and knowledge rather than an arbitrary one" (Pea 1987, 100). In other words, students must appreciate "the importance of participation in coming to know" (Brown and Walter 1983, 6).

Constructivist Theory.

These principles of learning and teaching reflect a shift from behaviorist to constructivist theory. The latter is growing in acceptance and in importance for mathematics education.

Another theoretical perspective that has permeated the mathematics education community is a very general form of constructivism in which it is acknowledged that students actively and personally construct their own knowledge rather than making mental copies of knowledge possessed and transmitted by teachers or textbooks. (Silver 1990, 7)

Constructivism underlies the new vision for mathematics education. "The Standards-Everybody Counts position has, for some researchers at least, coalesced into a very active concern to spell out, and analyze, the foundations of constructivism" (Davis, Maher and Noddings 1990, 2).
Constructivist theory is directed at student learning, but it also charts the path for teachers relearning about learning. Experienced teachers concerned about implementing the vision in the high school mathematics classroom must reconstruct their view of the learning process. Frustration with traditional practice implies disequilibrium. To reduce this state, teachers need to experiment with new approaches, gradually modifying them and incorporating them into their repertoire. The following targets attempt to focus this reconstruction.

The Targets and Options

This thesis clarifies the new vision of mathematics education by presenting an instructional framework of eight targets. Reflected are the two processes of instruction: learning and teaching. Learning mathematical power requires opportunities to form mathematical concepts and schemas, to engage in mathematical thinking, to communicate mathematical ideas and to develop a positive disposition toward mathematics. Teaching for mathematical power employs student-centered tasks, a variety of work formats, a range of mathematical tools and a choice of assessment alternatives.

The target sections include a description, a rationale and options. The description of each target contrasts the new and traditional models of instruction.
The rationale examines the target's relation to the requirements of the twenty-first century and to current learning research. Each target section also proposes options or suggestions for implementation. The tone of the thesis intentionally changes as the intended audience of classroom teachers is pushed to attempt implementation of the vision. The specific prescriptions given are only a taste of the possibilities and are presented to stimulate modifications and additional ideas.

Deep Understanding of Concepts and Schemas.

To achieve the vision, instruction must provide students with opportunities to construct deep understanding of mathematical concepts and schemas.

The teacher should demonstrate a deep understanding of concepts and principles, connections between concepts and procedures, connections across mathematical topics..., and connections between mathematics and other disciplines. (NCTM 1991, 89)

The content of mathematics instruction focuses on concepts and relationships rather than definitions and procedures. The emphasis is on depth not detail.

Deep understanding is based on the construction of knowledge. The emphasis shifts away from the traditional model with rote learning of facts and algorithms. Instruction moves students from the 'what is it' and 'how is it done' level of understanding to the 'why' level of ideas and generalizations (Davis 1978). Facts and
algorithms evolve from conceptual knowledge rather than act as prerequisites to it. "Students construct their understanding of mathematics by learning to use mathematics to make sense of their own experience" (California State Department of Education 1991, 28).

Rationale. Conceptual knowledge is the essence of the constructivist view and is necessary to meet the realities of the next century. Deep understanding of concepts and schemas is the goal of the learning principles sketched earlier. With understanding based on both why and how rather than solely on how, knowledge is more lasting and more adaptable (Skemp 1987).

Due to highly developed technology, an emphasis on paper and pencil calculation and symbol manipulation is outdated. The increasing complexity and expansion of the mathematical sciences make it impractical to 'cover' content. Understanding the structure of mathematics and the processes of acquiring that understanding is the content of the future.

Opportunities for developing deep understanding of concepts and schemas rest on learning principles. The first two options or prescriptions presented below attempt to implement assessment of prior knowledge and stimulation of disequilibrium. The third emphasizes restructuring content around concepts and schemas.
Derive instruction from the learner’s reality. To guide conceptualization, it is important to tap students’ existing knowledge and assumptions. The simplest way to do this is to ask. Introduce a ‘new’ topic by asking what students already know. In addition to examples, nonexamples and algorithms, probe for information about characteristics of a concept or the reasons behind a relationship.

Check information needed for a new topic through homework. Give the assignment: ‘Write examples of adding fractions that show all the important points. Include different types and be prepared to explain what each example shows.’ Discussion of the examples reveals understanding about deeper meaning.

Create disequilibrium. The learner restructures knowledge to accommodate new information, when disequilibrium is created. To clarify concepts, give nonexamples, as well as examples, to expand understanding (Crosswhite 1987). For instance, after identifying $x$ and $2x$ as like terms, ask about $y$, $x^2$ and $xy$. Let students hypothesize and test their ideas. The concept of like terms is refined in the process of examining the nonexamples.

Disequilibrium is created by introducing a new variation on a schema. After students are comfortable with using the quadratic formula to solve $x^2 + 5x + 6 = 0$, 

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present $x^2 + x = 6$. It is key that students do the work of adapting their existing knowledge to the new complication. When a teacher predigests information for the student, the student is only able to memorize the teacher's schema.

**Organize curriculum around concepts.** Drive curriculum and instruction from the perspective of a "big" idea (Cordeiro 1991) like equality, group theory, variables or functions, rather than by acquiring fragmented procedures associated with it. Students need to explore what a variable is beyond a rote definition. They need to examine characteristics and distinction of how it is used as well as how to solve for it.

Over a period of time, present an important concept as a theme. Chapter titles deserve attention as themes before related procedures are practiced. As examples and procedures are developed they can be related to the theme through brief discussions. In a more elaborate development of this approach, add to the curriculum special activities related to the concept theme. For example, a more complete understanding of functions develops by exploring real life phenomena modeled as equations. A series of experiences with a particular concept could be added to the curriculum as a short unit.

**Reflection.** The traditional model of instruction characterizes content as factual information and process as procedural method. In the new vision, the content is
conceptual knowledge. Concepts and schemas cannot be separated from the mathematical thinking that constructs them. "Content and process are being reconceptualized. They stand in relation to each other and each is embedded in the other" (Crowell 1989). This leads to the next target which explores the processes of mathematical thinking.

Mathematical Thinking.

Thinking is at the heart of the new vision of mathematics education; "mathematically powerful students think" (California State Department of Education 1991, 17). In the NCTM Standards two of the five student goals relate to thinking: "become mathematical problem solvers...[and]...reason mathematically" (NCTM 1989, 5). Also, NCTM standards at each grade level attend to mathematical thinking by focusing on reasoning, problem solving and making connections (NCTM 1989). "A major purpose of school mathematics is to develop in students the habits of thinking" (Silver 1990, 8). Some believe "the single most important reason to teach mathematics is that it is an ideal discipline for training students how to think" (Schoenfeld 1982, 32).

Mathematical thinking spans the entire range of cognitive skills and strategies. Traditionally, thinking in school mathematics involves rote memorization, application of algorithms and formal inductive and
deductive logic. However, the altered reality of the mathematical sciences requires other modes including "modeling, abstraction, optimization, logical analysis, inference from data and use of symbols" (National Research Council 1989, 31). With the vision emphasizing real applications, thinking skills and strategies are used in the content specific context of mathematics. "Mathematical thinking at its most powerful grows out of the kinds of thinking that are naturally part of everyone's repertoire" (California State Department of Education 1991, 18).

A distinction often is made between lower-order and higher-order thinking skills. Items on the National Assessment of Education Progress achievement tests characterize low level learning as knowledge and skill and high level learning as understanding and application (Peterson 1988).

For example, in a low level computation problem a student can look at the addition, subtraction, multiplication or division sign and know immediately what mathematical procedure must be performed to solve the problem. On the other hand, on a high-level mathematics problem the student first must figure out how to solve the problem. (10)

From the perspective of cognitive psychology, the distinction is made by comparing automatic and controlled information processing (Silver 1987). Lower-order skills require little conscious attention; whereas, higher-order skills require a student to control and often mediate, the processes used.
A complicating factor in distinguishing between higher-order and lower-order thinking skills is that what is a higher-order process at one stage may become a lower-order skill with practice. "In mathematics, much of the instruction given in arithmetic algorithms appears to be directed at automatizing the procedures of numerical computation that start out as controlled processes" (Silver 1987, 40).

Distinguishing higher-order and lower-order thinking helps define problem solving in terms of the new vision. Effective problem solving was a major theme of mathematics education during the 1980's. Unfortunately, classroom attention to this theme concentrated on short word problems. In the vision, such routine word problems exercising lower level thinking are regarded as exercises rather than true problems (California State Department of Education 1991). Problem solving entails challenging puzzles, guided discovery, investigations and long-range projects which require higher-order thinking skills and strategies.

The vision of mathematics education calls for the emphasis to shift from lower-order to higher-order thinking. Rote memorization and mindless algorithmic performance are displaced. Highlighted instead is higher-order thinking characterized as complex, non-algorithmic, judgmental, multi-faceted, indefinite, structure making, self-controlled and effortful (Resnick 1987). For the
purposes of this thesis, mathematical thinking and thinking
techniques and strategies in general will refer to higher-order
cognition.

Rationale. The emphasis on higher-order thinking in
mathematics education is based on the new learning research
and the challenges of the future. The constructivist view
conceives of knowledge built through experience and
reflection. Thus, thinking is the means to knowledge.
“Children will not succeed in learning maths unless they
are taught in ways that enable them to bring their
intelligence, rather than rote learning, into use for their
learning...[of]...mathematics” (Skemp 1987, 7).

Critiques of education call for an increased need for
higher-order thinking skills to improve performance
(McTighe and Schollenberger 1991). The need is echoed in
concerns about the mathematics preparation of students who
will live and work in the twenty-first century (National
Research Council 1989, Mathematical Sciences Education
Board and National Research Council 1990). Experience with
mathematical thinking “empowers us to understand better the
information-laden world in which we live” (National

Since the traditional view of the mathematics
classroom does not encourage higher-order thinking,
teachers need to create a classroom climate that encourages
this type of thinking. The following options help to
stimulate students’ mathematical thinking.
Use specific cognitive verbs. Specific cognitive verbs direct students to act using higher-order thinking skills. Directions to compare two strategies for solving a problem or to synthesize an algorithm given a series of worked-out problems prompt student thought more effectively than the general query 'what do you think about...?' Furthermore, such verbs indicate active, not passive, involvement on the part of the student.

Brainstorm. Brainstorming is a well-documented divergent thinking technique in which a group generates ideas without evaluation. When students brainstorm, criticism is deferred, the approach is freewheeling, quantity is the goal and combination and improvement are encouraged (Davis 1986). One thought stimulates other thoughts.

In the mathematics classroom, students can brainstorm observations of constructs, problem-solving strategies or possible questions for an upcoming test. The teacher records the ideas on the chalkboard, enforces the no judgement rule and encourages lots of ideas from all students.

Attribute listing is a variation of brainstorming in which attributes or characteristics of an object or idea are identified. Usually some organization of the attributes is structured before, during or after ideas are generated. For example, listing attributes of
parallelograms leads to categorization based on sides, angles and properties of symmetry. Also, problem solving strategies incorporate attribute listing in the problem posing (Brown and Walter 1983) or solution finding phases (Davis 1986).

Generate algorithms or models. Allow students to create an algorithm, a procedure or a model rather than use prepared strategies. Give students worked out examples and have them generalize a procedure for completing similar problems. Subsequent study of additional examples can help to refine the algorithm. Some teachers may be concerned about students adequately developing procedures required by the curriculum. However,

teachers soon discover that children are interested in the activities, and are naturally motivated by the creative possibilities of constructing their own models to fill the requirements of each problem. (Maher and Alston 1990, 161)

Ask students to develop a problem solving model. The process may begin with an unstructured list of miscellaneous suggestions gathered from group discussions, textbook pointers or individual experiences. Ideas can be accumulated on a posted sheet over a period of weeks. When the list becomes unwieldy, it is pruned and structured in categories. Eventually, categories are sequenced. In summary, a list, diagram or illustration synthesizes key aspects into a model of mathematical problem solving. The process can evolve from a whole group record to an
individual creation or vice versa. It is important that students synthesize the model, while the teacher structures and paces the lessons.

Instead of memorizing traditional content, students create their own knowledge that serves the same purpose. This reflects that with the realization of the new vision, the place of algorithms will be both diminished and enhanced--diminished in the area of memorizing algorithms for the purpose of turning out answers, but enhanced in the direction of learning to plan and design algorithms for human and computer execution. (House 1988, 4)

Reflection. A symbiotic relationship exists between the first two targets: deep understanding of concepts and schemas and mathematical thinking. Each requires the other. Concepts and schemas are the fuel of thought. Thinking produces concepts and schemas. "The growing alignment of mathematics learning with mathematical thinking is a significant shift in education" (Pea 1987, 90).

The essence of the vision is instruction for deep understanding and mathematical thinking. However, teachers need confirmation of such learning and thinking. The evidence emerges through communication about mathematics, the third target.

Communication about Mathematics.

The metaphor of mathematics as a language pervades the new vision. Some even propose that mathematics be
taught as a foreign language (Pimm 1987). This view highlights the need for students to communicate about mathematics.

The development of a student's power to use mathematics involves learning the signs, symbols, and terms of mathematics. This is best accomplished in situations in which students have an opportunity to read, write, and discuss ideas in which the use of the language of mathematics becomes natural. (NCTM 1989, 6)

"Mathematical power entails the capability to communicate about mathematics" (Mathematical Sciences Education Board and National Research Council 1990, 37).

Communication must not be restricted to "a lecture-oriented lesson or when students' responses are limited to short answers to lower-order questions" (NCTM 1991, 96).

Communication includes reading, writing, speaking and listening. Communication encompasses verbal and non-verbal modes. Students communicate through graphs, diagrams, flowcharts and models, as well as words. Informal mathematics communication is recognized as well as formal.

Instruction allows students to talk about their experiences and how they relate to mathematics concepts, to listen to each other as they share ideas, to read mathematics in various formats..., and to write about mathematical situations. (Phillips et al. 1991, v)

Communication about mathematics should span all aspects of learning mathematics.

Rationale. Communicating about mathematics benefits the student as a future citizen and a present learner. As society becomes more quantitatively dependent, there is
increased demand for skills related to communicating about mathematics. Those who can clarify and interpret mathematical ideas will be able to make and to persuade informed decisions.

In the classroom, communication allows the learner to clarify and refine mathematical ideas. When students express or represent mathematics, their knowledge solidifies. Two-way communication is important in the testing phase of concept development. Furthermore, communication is a vehicle for the teacher to assess students' learning.

The prescriptions below focus on written communication. This is done as writing demands student participation more than reading, speaking or listening. Also, writing can be done out of class or during class. The focus on writing is not meant to suggest that other modes of communication do not deserve attention.

Include writing on tests. Written definitions and completion questions are included easily on tests. Explanation and analysis, that better indicate understanding, can take the form of lists, short paragraphs or essays. "Once writing has been used as a testing tool, the verbs might include analyze, compare, contrast, explain, hypothesize, justify, read and explain, relate, restate, reword, summarize, support, suppose" (Azzolino 1990, 100). However, there is an important caution in
using this suggestion. When responses are memorized from the textbook, conceptual knowledge is not tapped and the prescription is empty of purpose.

Procedures like solving an equation can be explained in general or a question may ask for a procedure applied to a particular problem. The question may be structured to focus on one method or to elicit a preferred strategy from several choices. More open-ended responses are prompted by questions like 'State five important ideas about slope.'

When time is a factor, it is suggested that writing tasks be simplified (Azzolino 1990). Giving students possible essay questions before a test can minimize the pressure of time and aid review. Asking for a procedure as a list, such as how to graph a linear equation streamlines writing and aids organization.

Assign creative writing. Mathematics can be presented in forms usually associated with creative writing. A story elaborates a metaphor for operations like integer addition and subtraction. Simple rhymes, limericks and haiku can be created for mathematics. Dialogue for imaginary conversations can point out contrasts. For example, imagine an odd number talking to an even number or a dialogue on strength between a variable’s coefficient and its exponent. Fantasy can be grounded in mathematical knowledge.
Imagination is required when students create a situation requiring a particular use of mathematics. Word problems can be written for an equation type. For example, 'Write a problem that can be solved with a quadratic equation.' Or word problems can be written to fit a specific equation, like $2x + 3 = 23$. Unlabeled graphs are another source. Interesting scenarios can be created for graphs combining segments of lines with positive, negative, zero or undefined slope. Basic situations require elaboration to account for parameters and breaks in a graph.

Require student journals. Journals can be kept for a variety of purposes. However, in this section the suggestions assume that the purpose of a journal is reflective.

Entries can focus on student attitudes, questions, opinions and self analysis. Student attitudes are reflected by the completion of phrases like 'Algebra is useful in...' or 'I thought this lesson was...'. Students can be asked to pose questions in journals. These can range from requests for help to speculations for investigation. Critiques of outside readings can be done in journals. More than class discussion, journals give all students an opportunity to express their opinions. Journals are also effective for student self analysis. For example, ask students to write about their study patterns.
Evaluation and amount of structure are issues associated with journal writing. When students are unfamiliar with journal writing, structured assignments are suggested. Such structure can be reduced over time. Communication assumes a response, but reflective journals do not need a grade. Though a time burden, the teacher should read student journals and react to ideas and concerns. The journal can be an informal means of communication between teacher and student. Alternately, students can exchange journals and react to each others' thoughts.

Reflection. All modes of communication receive attention in the new vision of mathematics education. In reality, they are difficult to separate. When students exchange journals, reading results. When a student explains an exemplary test response to the class, there is speaking and listening.

The traditional perception of mathematics expands to encompass communication. As explored in the next target, other perceptions change in developing a positive disposition toward mathematics.

Positive Disposition toward Mathematics.

The vision of mathematical power is incomplete without opportunities for students to develop a positive disposition toward mathematics.
Students who have a positive disposition to mathematics are inclined to use mathematics to make sense of situations that come up in their lives; they use mathematics to achieve their own purposes. (California State Department of Education 1991, 22)

Mathematical power includes mathematical literacy plus a disposition to use one’s knowledge.

As discussed here, disposition encompasses attitude and perception. Disposition is reflected in two of the five student goals in the NCTM (1989) Standards: "becoming confident in one’s own ability" (12) and "learning to value mathematics" (13). The first pictures students working with self-confidence and perseverance.

In the long run, it is not the memorization of mathematical skills that is particularly important—without constant use skills fade rapidly—but the confidence that one knows how to find and use mathematical tools. (National Research Council 1989, 60)

Achievement is based more on effort and involvement rather than on mathematical ability reserved for a few.

The perception of mathematics is no longer limited to computation and preparation for careers in engineering and the physical sciences. The mathematical sciences are studied as

an exploratory, dynamic, evolving discipline rather than as a rigid, absolute closed body of laws to be memorized. They [students] will be encouraged to see mathematics as a science, not as a canon, and to recognize that mathematics is really about patterns and not merely numbers. (National Research Council, 1989, 84)

Above all, mathematics is valued by all students as useful preparation for employment and citizenship.
Rationale. Rapid technological change in recent decades has increased the importance of mathematics in our society. Technology also contributes to the continuing evolution of the mathematical sciences. Students must value mathematics and correctly perceive its nature to keep pace with society.

The feelings and perceptions of students affect their learning.

Many studies of classrooms have neglected to take into account pupil’s expectations, perceptions of school mathematics and interpretations of classroom events. Yet every teacher knows that these have a profound influence on classroom behavior and achievement. (Hoyles 1988, 147)

Attitudes of self-confidence and persistence lead to achievement in any subject not just mathematics. Students need to accurately perceive the nature of mathematics and how mathematics is learned in order to achieve. The options below aim to alter the traditional perception of mathematics.

Examine myths about mathematics. Many dysfunctional mathematical beliefs exist about the nature and learning of mathematics (Borasi 1990). Creating situations where students question these beliefs is productive. One approach is to present a belief and contradictory information, then let students react. The sidebars in Everybody Counts (National Research Council 1989) contain juxtaposed myth and reality statements that can be modified for classroom use. It is preferable that such discussions
be founded in students' experiences. For example, after several weeks of checking homework in small groups, students discuss the belief that mathematics is done individually. Students can compare the small group routine for homework with more traditional teacher-centered models.

The myth that only engineers and scientists need mathematics should be attacked proactively. It is important for students to perceive mathematics as valuable to their future. One approach is to explore the use of mathematics in various careers. Have students ask adults in a variety of occupations for formulas or calculations they use regularly. This can lead to the investigation of the reasoning and problem solving demands of different jobs. Interviewing adults in a variety of occupations points out the need for a broad view of the mathematical sciences.

**Introduce mathematical diversions.** Brain teasers, number tricks and puzzles tied to mathematical concepts reinforce a more positive perception of mathematics and stimulate interest in learning. Regular assignment of puzzles develops a variety of reasoning strategies. Number tricks are explored for the underlying mathematical principles. Paradoxes and unsolvable problems help challenge the view students hold of mathematics as having only one right answer. Egyptian multiplication and other alternate computational methods introduce a cross cultural
perspective. Famous puzzles, like The Seven Bridges of Konigsburg, highlight the history of mathematics. Lesser known fields of mathematics can also be introduced with puzzles. For example, coloring a one-sided Moebius strip relates to topology.

The rationale is not only the motivational quality of fun. In fact, recent critiques suggest that, due to the complexity and subtlety of motivational research, a little information is dangerous. "Trying to make learning always fun is impossible and creates a counterproductive mindset in students" (Willis 1991, 4). Students react negatively to mathematics instruction which does not fit their perceptions (Borasi 1990). Therefore, it is important to use mathematical diversions for a purpose. Some diversions illustrate specific concepts or relationships. Others provide a setting for problem solving. Whatever the purpose, communicate it to students.

Reflection. It is socially acceptable to proclaim incompetence in mathematics. "Only in America do adults proclaim their ignorance of mathematics (‘I never was very good at math’) as if it were some sort of merit badge" (National Research Council 1989, 76). Mathematical illiteracy carries none of the shame of verbal illiteracy. Unfortunately, as children become socialized by school and society, they begin to view mathematics as a rigid system of externally dictated rules governed by standards of accuracy, speed, and memory. Their view of mathematics shifts gradually from enthusiasm to apprehension, from confidence to fear. Eventually, most students leave mathematics under duress, convinced
that only geniuses can learn it. Later, as parents, they pass this conviction on to their children. Some even become teachers and convey this attitude to their students. (National Research Council 1989, 44)

Teachers need the support of other adults to break this cycle. Implementing the vision of mathematics education includes changing these dispositions about the need for mathematics, the nature of mathematics and the learning of mathematics.

Change in traditional written curriculum reflects implementation of the next four targets: student-centered tasks, collaborative work format, mathematical tools and alternative assessment. In contrast to the more elusive behavior targets discussed above, these four targets emphasize pedagogical change. All the targets are inseparably interwoven; the curriculum targets are the woof to the warp of student mathematical power.

Student-Centered Tasks.

A curriculum for mathematical power must be built on a core of mathematical tasks or activities involving extended problem solving, investigation and application. In the traditional model, the textbook dictates a curriculum in which mathematics is presented by the teacher in manageable chunks for student absorption. In the new vision of mathematics education, the focus shifts from the textbook to tasks rich in mathematics and context through which students learn by doing.
Teachers should choose and develop tasks that are likely to promote the development of students' understandings of concepts and procedures in a way that also foster their ability to solve problems and to reason and communicate mathematically. (NCTM 1991, 25)

It is desirable to include activities that incorporate all the targets for developing mathematical power.

In this section the focus is on student-centered tasks as distinguished from traditional exercises and word problems. The purpose of the latter is for students to master an isolated topic by practicing procedural skills. There is a place for procedural skills in the vision, but "the richer tasks subsume the routine work and make it possible for students to demonstrate the full range of mathematical work instead of focusing on its components" (California State Department of Education 1991, 17).

The goal of a task should be to involve students in exploration and reflection that results in deeper understanding of mathematics and its contextual use. Tasks may be investigations, projects, applications or extended problems. In all cases they are student-centered requiring active participation by students.

In discussions of the identification or creation of tasks, four characteristics represent common themes. Good tasks incorporate a rich context, actively engage students, are open-ended and provide an opportunity to mathematize. Tasks with a rich context are authentic, tap students' prior knowledge and generate higher-order thought. Meaningful tasks "are such that the context is
kept active in the reasoning process (as a meaningful
guideline)" (Janvier 1990, 190).

Tasks require active student participation and engage
students. "Students are the workers and the decision
makers. Students interact with other students" (Stenmark
1991, 16). Student involvement is not passive as in the
traditional model of instruction. Active participation is
required. Good tasks also engage students; they display
interest and persistence. Not only is interest and
stimulated by "tasks that relate to the familiar everyday
worlds of the students; theoretical or fanciful tasks that
challenge students intellectually are also interesting"
(NCTM 1991, 27). Even "games can be effective teaching
tools" (Bright, Harvey and Wheeler 1985, 122) at many
cognitive levels.

Student-centered tasks allow for a variety of
approaches and often different solutions. Educational
research supports the open-ended nature of tasks.
"Students are often able to learn usable knowledge and
skills more effectively and efficiently through experience
with non-goal-specific problems and exercises than with
more traditional goal-specific versions" (Silver 1990, 4).
Open-ended tasks require making more connections between
important ideas which improve the understanding and
organization of knowledge.

Good tasks are structured so that students experience
the value of mathematics while dealing with the given
situation. This is "a result of 'mathematizing', which is
the process of organizing experience in ways that are
distinctly mathematical" (Steffe 1990a, 45). This goes
beyond the demonstration of mathematics. It means that the
task is open-ended enough so students can individualize
and, thus, own the method of using mathematics. Students
find mathematics an effective method for dealing with the
task.

Rationale. A typical word problem from a mathematics
textbook "is a classic case of a bit of nonsense
masquerading as an 'application' of mathematics"
(Willoughby 1990, 14). In contrast, tasks reflect the way
mathematics is used in the real world. "Mathematicians
analyze problems and create algorithms, they do not merely
memorize algorithms and recall them as needed" (Davis and
Maher 1990, 77). Thus, tasks prepare students for a future
in which they will need to use mathematics in complex, real
situations rather than in contrived, simplified exercises
and word problems.

Learning through active engagement in tasks is
consistent with constructivist theory. For students to
develop deep understanding, direct interaction with
mathematics is necessary. "Meaningful learning occurs when
children are actively engaged intellectually and
emotionally" (Baroody and Ginsburg 1990, 57).
Teachers who commit to student-centered tasks must set aside traditional views of efficient delivery of instruction. Adequate time is crucial to build student interest. Open-ended tasks based on rich contexts take time to explore. Sufficient time needs to be allocated for piloting the options below.

Use real raw data. Collect real data for students to analyze. Ask students to present findings on sport team statistics, school attendance data or final exam errors. Students can generate their own data through surveys; fieldwork should be part of mathematics. Other teachers and administrators may provide data and be grateful that a real problem can be analyzed. When the data originates close to the student’s real life, the context is realistic and engaging.

Give students raw data. The more complex and messier the better. Do not simplify or filter the data except to protect confidentiality. To do so deprives students of an opportunity to do mathematics. Computers and calculators make the difficult easy and the infeasible possible.... As a consequence, students can solve realistic problems that are relevant to their everyday experiences and that have the potential of stimulating continuing interest in mathematics. (Mathematical Sciences Education Board and National Research Council 1990, 20)

Key to the use of such data is that students design the analysis. Create an opportunity to mathematize, by asking students to represent, support and communicate conclusions about class grades, school absenteeism or
school lunch preferences. Furthermore, student interaction about various possibilities can result in better decision making.

**Reflection.** Active, engaging mathematical tasks are the superstructure for the new mathematics curriculum. They replace the chapter section as the organizing unit of classroom instruction.

The effectiveness of each student-centered task depends on its use. "Delivery of instruction is inseparable from curricular content" (California State Department of Education 1985, 12). If a teacher directs students through a task, the task does not serve the purpose for which it was designed. To use tasks properly, teachers and students must use various work formats, all of which actively involve the students.

**Variety of Work Formats.**

To develop mathematical power, a curriculum must allow students to work independently and collaboratively.

Classroom structures...are varied: students may at times work independently, conferring with others as necessary; at other times students may work in pairs or in small groups. Whole-class discussions are yet another profitable format. No single arrangement will work at all times; teachers should use these arrangements flexibly to pursue their goals. (NCTM 1991, 58)

The traditional model of instruction centers on students working in isolation whether as individuals or within a whole class group. In working toward the vision, teachers
need to expand the role of collaboration in the high school classroom.

Currently, attempts to develop collaboration focus on cooperative or small group learning. Small groups serve a variety of instructional purposes including discussion, discovery, problem solving, modeling, drill and review. Yet, "simply placing students in groups and telling them to work together does not in and of itself promote greater understanding of mathematical principles and ability to communicate one's mathematical reasoning to others" (Johnson and Johnson 1990, 104-105).

In addition to student interaction centered on a meaningful task, small group cooperative learning has two key characteristics: positive interdependence and individual accountability (Slavin 1989/1990). Interdependence is established by splitting resources or information among group members, by requiring a single group product or by giving group rewards. Individuals are accountable on individual tests, through random answering and through the results of feedback on group functioning. "The classroom becomes a community of learners, actively working together in small groups to enhance each person's mathematical knowledge, proficiency and enjoyment" (Davidson 1990a, 1).

Rationale. Today the ability to work collaboratively is an economic asset. Employment forecasts identify the
importance of interpersonal skills and teamwork at all job levels (Packer 1992). Economic comparisons with Japan lead corporate leaders to embrace the principles of the total quality movement which foster commitment and collaboration rather than competition among workers (Bonstingl 1992).

Learning research supports the idea that learning is a social act. Knowledge is constructed in a social context through questioning, discussion and debate. These techniques reinforce the vision of mathematics classrooms as mathematical communities "directed not solely toward the acquisition of the content of mathematics in the form of concepts and procedures but also toward the situated, collaborative practice of mathematical thinking..." (Silver 1990, 9). Mathematical communities are carefully built from collaborative experiences like those proposed in the following options.

Employ think-pair-share. Teachers committed to introducing collaboration in the traditional classroom are advised to start slowly. Administrators, parents and, most importantly, students need time to adjust to the new format. One suggestion is to use think-pair-share (Lyman 1981). The class groups in pairs and the teacher poses a question. Students think individually about a reply for a period of time. Next, students pair with their partners, discuss the problem and agree on a response. Lastly, students share their answers with the rest of the class.
Think-pair-share is simple and applies to many situations. The method is easy to incorporate into the traditional mode of instruction. Interjecting think-pair-share into lecture presentations tests students' understanding and wakes up students having difficulty staying attentive (Robertson, Graves and Tuck 1990). Another logical use of the think-pair-share technique is for review. Pairs force participation from all students. Testing a response with a partner promotes the self-confidence necessary to answer and to defend the answer in a larger group. From this method, teachers move easily to more complex structures.

Use homework groups. Formation of homework groups using cooperative learning principles is another setting in which to introduce the small group format. Students establish the routine of starting the class by reviewing homework in groups of four. "Students learn by explaining an answer or explaining why an answer is incorrect to another student or by helping other students with their work" (Peterson 1988, 15). Only when all in the group request help with a problem does the teacher intervene. At the end of the allotted time, the teacher randomly chooses one member of the group to submit work for the entire group. As needed the teacher leads a class discussion clarifying confusing material. The start of class becomes
automatic and groups are able to focus on individual difficulties.

Application of cooperative learning principles in the formation of homework groups leads to success. "In mathematics classes, groups with four members seem to work best" (Davidson 1990b, 56). Randomly formed or heterogeneous groups function best (Robertson, Graves and Tuck 1990). Create random groups by drawing cards with mathematical symbols. All radical bars form a group, all equal signs form a group and so on. An easy way to form teacher-selected groups is to arrange a set of recent tests by score from high to low. Start a pile for each group with one of the highest scoring tests. Place a low scoring test from the bottom of the pile in each group. Continue so that each pile contains tests with high medium and low grades. Review the group piles and make adjustments to reflect the heterogeneity of the class. Group roles are not assigned, as research indicates that students abandon or switch assigned roles (Good et al. 1989/1990). However, include in each group a task master, a student who will attempt to keep the group on task.

Homework grouping pays attention to group process. Introduction of the groups includes formulating general guidelines for effective cooperation. Subsequent discussion of group functioning occurs if needed. Every four to five weeks, groups are changed. Before each
change, students evaluate how the group functioned in writing or in a whole class or small group discussion.

Reflection. Students and parents, as well as teachers, need to adjust to the cooperative working mode. Students arrive in the classroom with "years of independent and competitive lessons to unlearn" (Schultz 1989/1990, 43). For some students and parents, small group cooperative learning is a threat to the valued position of individual excellence and success through competition. Yet, outside the classroom, mathematics is rarely done in isolation. A recent study offers a collection of conversations with and about contemporary figures in mathematics. It is surprising how frequently the mathematicians mention collaboration, both in terms of influences on their work and in terms of the benefit they derive from working with others. (Schoenfeld 1987, 211)

An understanding of how mathematics is used outside the classroom also expands the current perception of the next target: mathematical tools.

Mathematical Tools.

In the new vision of mathematics education, "tools" mean more than pencil, notebook, textbook and, possibly, calculator. Tools can be either literal, like manipulatives, calculators and computers or figurative, like conventional notation, procedural algorithms and problem solving heuristics. Figurative tools are "the classic intellectual tools and techniques of mathematics"
(California State Department of Education 1991, 20), but are now conceptualized as aids to doing mathematics, not the essence of mathematics. The concept of mathematical tools needs restructuring to include long established notation and algorithms as well as manipulatives and new technologies.

A curriculum for mathematical power must provide a range of mathematical tools including technology.

Teachers must value and encourage the use of a variety of tools rather than placing excessive emphasis on conventional mathematical symbols.... Teachers should help students learn to use calculators, computers and other technological devices as tools for mathematical discourse. (NCTM 1991, 52)

Reliance on paper and pencil calculation and manipulation should be replaced with estimation and mental calculation, as well as computers and calculators.

Calculator and computer use attracts attention due to the rapid technological advances of the last two decades. However, "the public at large has gotten the strange notion that using a calculator in school is somehow cheating" (Willoughby 1990, 62) and that calculator use will hinder mastery of essential skills. Research contradicts this belief with evidence that use of calculators "is not likely to obstruct achievement of skill in traditional arithmetic procedures" (Mathematical Sciences Education Board and National Research Council 1990, 23).

On the other hand, there is public support for computers in schools. This exists even though research at
the high school level shows that barriers to the actual use of computers still outweigh incentives (Schofield and Verban 1988). Also, software programs do not develop mathematical thinking, since "the majority of instructional programs (more than 90 percent by one reliable estimate) have as their goal to train the user in some particular low-level skill" (Willoughby 1990, 67).

Despite these limitations, calculators, computers and manipulatives are tools for doing mathematics in the new vision. Manipulative materials help students internalize concepts before moving to abstract representation. Calculators and computers allow students to work with raw data without getting mired in computation. Computers free students for higher-order thinking like exploring relationships, testing hypotheses and problem solving. With a wider range of tools, students learn to assess when, as well as which, tools are appropriate. "The ability to select appropriate tools and techniques and to use them effectively is an essential part of mathematical power" (California State Department of Education 1991, 20).

Rationale. A future with technology requires expanded use of mathematical tools. Simple calculators are more efficient at arithmetic computations than paper and pencil. In fact, today there exist "calculators and computers costing less that $100 that can perform most of the mathematical symbol manipulation taught in schools
between kindergarten and the second year of calculus” (Willoughby 1990, 60). Advances in computer graphics are also expanding the applications of technology. To prepare for the future, students need to learn how to use, not necessarily understand, this technology.

Mathematical tools enhance conceptual learning and high level mathematics. This goes beyond the use of computers as "high-tech flash cards" (Mathematical Sciences Education Board and National Research Council 1990, 18). Understanding is deepened by representation of concepts at the concrete level (manipulatives) and pictorial level (calculator or computer graphics). The need continues into high school as Piaget’s formal operational stage often is not reached until adolescence (Resnick and Ford 1981).

Also, computers and graphing calculators allow students to create algebraic and geometric representations of a concept or problem. Such multiple representations are important in learning mathematics (Demana and Waits 1990). Evidence even exists that algorithms can be learned more quickly when concepts are developed first with software than when instruction is traditional (Mathematical Sciences Education Board and National Research Council 1990).

Some research suggests that technology aids in the acquisition of higher-order skills. Students taught with calculators show better performance in problem solving on standardized tests than students not using calculators. "In particular students using calculators seem better able..."
to focus on correct analysis of problem situations" (Mathematical Sciences Education Board and National Research Council 1990, 23). Successful programs based on computers and calculators exist in which an understanding of mathematical concepts and problem solving ability are developed before the conventional skills and mastered (Heid, Sheets and Matras 1990; Demana and Waits 1990).

The following prescriptions use tools that have received pedagogical attention in the last decade: manipulatives and technology.

**Develop a concept with manipulatives.** Despite association with the elementary level, manipulative materials are appropriate for secondary mathematics instruction.

Some middle- and high-school students believe that they are too old to use manipulatives—these materials are for the 'little kids'. This resistance will persist until the students have opportunities to explore 'advanced' concepts with concrete materials. (California State Department of Education 1991, 46)

Geometric models solidify characteristics and properties. Colored chips are used to explore integer operations. Algebra tiles model simplification of algebraic expressions. Whether real like random selection or contrived like the Tower of Brahma, manipulatives are useful in modeling problem situations. Hands-on manipulatives serve as conceptual reference points and help to model operations and procedures.
When using manipulatives, follow the cycle of exploration based in learning research (Resnick and Ford 1981). A period of free play is necessary for students to feel comfortable with manipulatives. This is followed by structured use. At this point symbolic representation is introduced to record results. At a latter stage, students extract rules for themselves. For example, students are given algebra tiles. After time to examine and conjecture, the factoring of trinomials is modeled and students progress from tile drawings to algebraic notation. Finally, students devise strategies for modeling other operations.

Availability should not hinder use of manipulatives. Teachers can order manipulatives from educational publishing companies or can adapt everyday materials. Use bingo markers for colored chips. Prepared patterns allow students to construct geometric solids. Students can construct more elaborate manipulatives such as a balance beam. After some experience with manipulatives, students may attempt to design a manipulative relating to a particular concept or procedure.

Explore concepts with graphing technology. Graphics are available for algebra, geometry and data analysis. Graphing utilities allow students to compare and relate visual and algebraic representations of functions. Students conjecture and test with software designed to illustrate
geometric hypotheses. Students visually explore data translated to graphs.

Powerful software combining a spreadsheet and a graphing utility is strongly recommended. Though a teacher will need to invest time in learning the software, the benefit is that connections between tables of values, algebraic statements and visual representations can be explored. Such software is also a powerful problem solving tool. Graphing data organized in a spreadsheet aids in discovering an algebraic rule. Data analysis uses technology to plot data points and find a regression line. Students are then free to make predictions about meaningful problems.

Reflection. The use of technology in the classroom must avoid the pitfall of becoming an end rather than a tool for doing mathematics. Just as traditional algorithmic tools do not represent mathematical understanding, access to this technology is no guarantee that any student will become mathematically literate. Calculators and computers for users of mathematics, like word processors for writers, are tools that simplify, but do not accomplish, the work at hand. (NCTM 1989, 8)

Researchers and educators who write about technology in the classroom underscore the rapidity with which change will continue. Interactive computer instruction, remote classrooms and computer access to immense databases are sprouting. "The prospective changes could bring a new
order of priority among traditional topics, new mathematical ideas, and new approaches to teaching and assessment of student learning" (Fey 1984, 6). The latter is explored in the next target.

Assessment Alternatives.

To many high school mathematics teachers, evaluation and assessment are interchangeable. However, distinguishing between them helps clarify the new vision of mathematics education. Consider evaluation as determining value; this implies ranking. In this view, evaluation refers to assigning grades, to placement in accelerated and remedial programs and to comparisons based on standardized tests. In contrast, assessment is defined as determining the current state of knowledge or skill. Assessment means gathering information for instructional decisions. Assessment subsumes traditional evaluation for grades and placement.

A curriculum for mathematical power must provide assessment at all stages of instruction. In the new vision assessment is an everyday activity which supports learning and teaching. "Mathematics teachers should monitor students' learning on an ongoing basis in order to assess and adjust their teaching" (NCTM 1991, 63). From this perspective the purpose of a test or quiz is to "shape and guide instruction and not to remain separate from it"
Mathematical power is a blend of conceptual knowledge, skill and disposition. Consequently, assessment of all dimensions of mathematical power influences instruction.

They [teachers] should assess students' understandings of concepts and procedures, including the connections they make among various concepts and procedures. Teachers must also assess the ability of students to reason mathematically—to make conjectures, to justify and revise claims on the basis of mathematical evidence and to analyze and solve problems. Students' dispositions toward mathematics—their confidence, interest, enjoyment, and perseverance—are yet another key dimension that teachers should monitor. (NCTM 1991, 63)

Assessment shifts from traditional content to all aspects of the process of doing mathematics.

Both performance assessment and authentic assessment are emphasized. In performance assessment the student demonstrates a specific behavior targeted for assessment. The demonstration takes the form of actual performance or is based on a product. Authentic assessment is performance assessment with the added criteria that the performance is done “in a real-life context” (Diez and Moon 1992, 40).

"Multiple choice questions may have a place in mathematics assessment, but they are inadequate for our new goals" (Stenmark 1991, 6).

Given the range of content and process to be assessed and given student differences, the use of a variety of assessment methods is indicated. Teacher observation,
interviews, journals, reports on investigations, essays, whole class discussion, oral reports, extended projects and portfolios join tests, quizzes and homework to complete the new vision of assessment.

Rationale. In the traditional model of instruction, "testing to assign grades is one of the most common forms of evaluation" (NCTM 1989, 203). High school evaluations in mathematics "focus on narrow skills and rote recall of information" (Leinwand 1992, 3). However, outside of school, assessment is based on tangible products or demonstrated skill. Business, labor and government agencies recommend performance testing, portfolio reviews and project evaluations as the basis for certification in high performance work skills (Packer 1992).

The constructivist learning research reinforces instruction as the goal of assessment. Constructivism "requires that instruction build upon children's existing knowledge" (Baroody and Ginsburg 1990, 63). Informal and formal assessment provide information for structuring learning. Thus, there is a continuing cycle of assessment followed by instructional decision making.

Assessment sends a strong message about what is important. "It is through our assessment that we communicate most clearly to students those activities and learning outcomes that we value" (Clarke, Clarke and Lovitt
1990, 128). If the vision is to be implemented, all dimensions of mathematical power must be assessed.

In an instructional environment that demands a deeper understanding of mathematics, testing instruments that call for only the identification of single correct responses no longer suffice. Instead, our instruments must reflect the scope and intent of our instructional program to have students solve problems, reason and communicate. (NCTM 1989, 192)

Assessment must reflect the goals of mathematical power. Though to have a major impact on assessment methods requires the support and commitment of many throughout the school community, a classroom teacher may move in this direction with simple modifications like those suggested in the options below.

Modify quizzes and tests. Construct a test that requires, not just allows, use of a calculator. This adds a small element of authenticity and validates the use of the calculator as a mathematical tool. Ask students to write about how they would solve a problem, rather than to actually do the calculation or symbol manipulation. This places the "emphasis on the process of problem solving" (Ferrucci and Carter 1992, 25) not the final result.

Include questions which allow for a variety of responses. For example, given an equation, write a problem the equation could solve or given multiple interpretations of data, justify one. Design questions to assess all the aspects of mathematical power. Such changes allow the classroom teacher to act as an agent of change without waiting for support outside the classroom.
Change grading practices. Reexamining the what, how and who of grading are also steps toward the new vision of mathematics education. For instance, decide not to assess every piece of work as suggested by the New Zealand Department of Education and quoted by Stenmark.

It is very easy to think that every piece of work that the students carry out should be assessed. This dramatically increases the teacher’s workload and the student’s stress level, and it does not necessarily produce a more effective assessment of students than can be achieved by carrying out a more selective assessment programme. (1991, 15)

For example, exploratory or practice tasks need not be graded. Help students develop intrinsic motivation rather than motivation based on a grade.

Assess student work with a reaction, rather than grade. Written or verbal comments respond to work with more information than a naked grade. By writing back and forth in student journals teacher and student develop a rapport which supports the learning process. When grades are required, tie points to descriptive statements. For example, phrases like "No work shown...Pertinent facts shown with inappropriate procedure...Clear and appropriate plan...Error in calculation...[and]...Correct answer" (Szetela 1987, 37) are matched with increasing scores.

Expand the 'by whom' of assessment to have students "help to create and apply standards for quality work" (California State Department of Education 1991, 54). Involvement in criteria setting facilitates student understanding and acceptance of assessment.
This builds ownership of the evaluation, makes it clear that judgments need not be arbitrary, and makes it possible to hold students to higher standards because criteria are clear and reasonable. (Wiggins 1992, 30)

Student set criteria are very helpful when introducing new formats for assessment such as extended projects or artwork. Students exchange feedback through discussion of and written reactions to each other's work. For example, after an oral presentation, each student in the class makes brief comments on slips of paper. These are then shared with the presenting student. In addition, teachers can make self-assessment a goal. Have students provide a written evaluation as part of project work. This can be open-ended or can target certain criteria. "Self assessment promotes metacognitive skills, ownership of learning, and independence of thought" (Stenmark 1991, 55).

Reflection. The specifics of a particular course fade before the memory of the final grade.

Psychologically, grades are the culmination of an educational experience, answering the need to know 'How well did I do?' Practically, they act as passports to the next step—the next grade level, the new school, the new job. (Stenmark 1991, 50).

Teachers are caught in the trap of desiring more meaningful assessment, yet recognize the realities of student class rank and grade prerequisites for courses (Maher and Alston 1990). Students, teachers, administrators and parents must be involved in reconceptualizing 'grades'. Thus the new vision redefines assessment and the other targets in order
to educate for mathematical power. In the process, the roles of teacher and student are redefined.

**Teacher and Student Roles**

The new vision of mathematics education launches teachers and students into new roles.

Teachers must guide, listen, question, discuss, clarify, and create an environment in which students become active learners who explore, investigate, validate, discuss, represent, and construct mathematics. (Howden 1990, 21)

If the vision of student-centered instruction is to replace the traditional model, teachers must relinquish control of learning and students must accept responsibility for it.

A change in classroom action reflects alterations in the teacher's role. No longer is the teacher lecturing to rows of silent students or fielding questions while students work individually. Most often, but not always, the teacher moves among groups of students who work collaboratively. Dialogue is two-way between teacher and student (and between student and student). The teacher probes rather than answers and stimulates rather than prepackages thought.

The redefined teacher role involves subtle changes as well. "The traditional teacher roles of authority figure and information disseminator must change to learning facilitator and instructional decision maker" (Phillips et al. 1991, vi). The basis of teacher authority shifts from
source of disciplinary control to the source of knowledge. The teacher becomes one of many resources for the student, not "the sage on stage" (Willoughby 1990, 94). Teachers have always made instructional decisions, but in the new vision this is a continuous function. The teacher changes methods in response to ongoing student assessment rather than working through a preset plan.

Teaching and learning are reciprocal processes; an alteration in one evokes an adjustment in the other. If the teacher no longer accepts the role of complete authority, then the student must be more accountable. If the teacher no longer organizes and presents all knowledge to be learned, then students must construct some for themselves. In the traditional model, students know the teacher will "tell them the basic principles, so why should they bother with the hard work that is involved in doing mathematics or with even reading the textbook?" (Steffe 1990a, 43). In the vision for mathematics education, the student accepts responsibility for learning.

"Like their teachers (students) will need 'staff' development in these new ways of working and studying" (California State Department of Education 1991, 11). Students need to accept "the concept of students as interpreters of their experience rather than absorbers of knowledge" (Wenger 1987, 219). The path to knowing mathematics is doing mathematics. The student's role includes validating, investigating, exploring,
constructing, discussing and representing. It is active, not passive. Students, also, need to accept the challenge of self-assessment and to become self-directed learners.

Experiences designed to foster continued intellectual curiosity and increasing independence should encourage students to become self-directed learners who routinely engage in constructing, symbolizing, applying and generalizing mathematical ideas. (NCTM 1989, 128)

For teacher and student the goal is to make the student independent of the teacher.

Focus on Instruction over Content

At this point it is appropriate to ask, "What about content?" "Doesn't the vision call for addition of statistics, probability and discrete mathematics to the high school curriculum?" "Shouldn't algebra be required of all students?" "Isn't what is learned as important as how it is learned?"

The answers to all these questions are in the affirmative. However, given the audience to which this thesis is addressed, the emphasis is on the process of instruction not on content. The experienced mathematics teacher struggling to understand and implement the vision is burdened by years of learning and teaching in the traditional model of instruction. Experienced teachers concerned about implementing the vision need to restructure their conception of learning and teaching. New content
taught with traditional methods will not effect mathematical power.

The conviction that experienced classroom teachers need to focus on restructuring instructional process rather than content does not mean to imply that content and process are dichotomous. The relationship is better characterized as symbiotic. The vision foresees the classroom teacher with an instructional repertoire expanded significantly beyond the traditional model. With more options, the teacher will be able to select the process that best serves the topic at hand. Content should influence the choice of instructional method.

Though this thesis recognizes that content and process are interdependent, priority is given to instructional process in an attempt to meet the needs of the experienced teacher. Thus, for the classroom teacher, a new model of instruction is the focus of restructuring for mathematical power. However, researchers report that "mathematics teachers find it very difficult to change their teaching" (Steffe 1990b, 167). To redesign instruction, teachers need a strategy for redesigning their own knowledge about learning and teaching. Focusing instruction on mathematical thinking is the strategy proposed in the next chapter.
CHAPTER III
RESTRUCTURING BY TEACHING THINKING

Overview

The previous chapter clarifies the vision of mathematics education in a framework of instructional targets. This chapter proposes a strategy for experienced high school mathematics teachers to restructure their teaching and thus implement the framework. Specific examples of this restructuring strategy are the topic of the next chapter.

This thesis recommends focusing on the teaching of thinking as an implementation strategy for experienced teachers struggling with restructuring mathematics. The chapter begins with a review of the difficulties encountered by this group and an explanation of how the proposed strategy responds to these challenges. Further arguments for this approach include the importance of mathematical thinking to the vision and the richness of the existing cognitive education movement.

The next section orients the mathematics teacher to thinking skills and strategies. A brief discussion of major aspects of the thinking process leads to identification of thinking skills and strategies associated with mathematics.
The subsequent three sections address thinking instruction. Approaches to planning thinking curricula are classified at three levels. Their relation to this thesis is explained. Key issues related to teaching thinking are reviewed. This leads to a five part structure for an effective lesson on thinking.

The chapter comes full circle by returning to the framework of targets. A review of the instructional targets demonstrates that they are embedded in thinking instruction.

In conclusion, Perkins’ (1986) concept of “knowledge as design” is revealed as the organizing structure for the framework of targets, the lesson plan and the techniques presented in the next chapter.

Teaching Thinking as a Restructuring Strategy

Restructuring for the Experienced Teacher.

Challenge of change. The experienced teacher possesses well developed schemas about teaching mathematics. Comparative research on expert and novice mathematics teachers confirms that “this storehouse of information that experienced teachers have accumulated about students appears to enable them to characterize what kinds of learning and behavior problems they can expect” (Berliner et al. 1988, 89). This integrated knowledge underlies the teacher’s beliefs and behavior.
Unfortunately, the schema of the experienced teacher can work against implementation of the new vision of mathematics education. The traditional model of instruction dominated the classrooms in which today's experienced teacher was a student. The traditional model set the standard of teaching throughout most of the experienced teacher's professional life. However, as previously discussed, the traditional view and the new vision of mathematics education conceptualize the teacher's role differently. In attempting restructuring, the teacher must recognize the likelihood that the new model of instruction conflicts with the background and practice of experience. Thus, the experienced teacher faces the challenge of setting aside unproductive habits in attempting to implement the new vision of mathematics education.

A strategy for restructuring. The strategy recommended to the experienced teacher in this situation is to focus on teaching thinking. This strategy approaches mathematics as an opportunity for students to actively engage in thinking. The context for student thinking is real-life application of mathematics to develop mathematical concepts. The priority for restructuring is to create instruction that develops students' mathematical thinking skills and strategies.
New role for teacher. By making teaching for thinking a priority, the backdrop for change shifts. Instead of trying to attack the monolith of traditional instruction, this strategy develops new ideas and behaviors which gradually replace or supplement traditional methods.

With this strategy the experienced teacher builds schemas and practices techniques related to teaching thinking. However, disequilibrium is created as the model for teaching thinking is different from the traditional model of instruction.

Teaching for thinking should not be viewed as simply adding another subject matter or set of skills that we teach in the same old way. Rather, teaching for thinking calls for a transformation of all our instruction and should be infused throughout it. It is only through well-designed classroom structures and a redefined role of the teacher in the classroom...that thinking will be promoted. (Costa and Lowery 1989, xii)

As the classroom teacher becomes experienced in teaching thinking, aspects of the traditional role which are inconsistent with the new vision weaken. If teaching thinking only reinforces the traditional model of instruction, the strategy is useless, if not detrimental, in implementing the vision.

Distinctions in teaching thinking. Costa (1991c) distinguishes among teaching for, teaching of and teaching about thinking. Teaching for thinking involves creating situations where students are allowed and encouraged to actively engage in thinking. Teaching of thinking implies students receive direct instruction in thinking skills and
strategies. Teaching about thinking makes students aware of their own and others' use of thinking in real-life applications.

If limited to the teaching of thinking, the traditional model of instruction is reinforced. This is not what is required to implement the new vision of mathematics education. The strategy recommended here emphasizes teaching for thinking and teaching about one's own thinking in the context of mathematics.

The strategy of focusing on teaching thinking provides a new perspective which enables the experienced teacher to move beyond the traditional role. However, teaching thinking promotes implementation of the new vision in other ways. Two aspects will be explained in the next sections.

Thinking in the New Vision.

Since there is little point in focusing on a topic not important in the new vision of mathematics education, one asks if mathematical thinking is significant in restructuring mathematics. The answer is an emphatic yes. Thinking is an important goal of mathematics instruction, a key in learning in any area and a desirable skill for future citizens.

Reformers point to the need for higher-order thinking. "A flurry of high-level government-sponsored reports has indicated the thinking skills in children have
reached an abysmally low level" (Baron and Sternberg 1987, ix-x). In response mathematical thinking is one of the key targets of the instructional framework presented in Chapter II. "Our top priority should be the development of students' thinking and understanding" (California State Department of Education 1991, 43). Thus, the vision emphasizes students actively using a variety of thinking skills and strategies in the mathematics classroom. 

Thinking about mathematics is a means as well as an end. Students think in order to understand.

Knowledge, by its very nature, depends on thought. Knowledge is produced by thought, analyzed by thought, comprehended by thought, organized, evaluated, maintained, and transformed by thought. Knowledge exists, properly speaking only in minds that have comprehended and justified it through thought. (Paul 1990, 46)

The process of thinking is essential to deep conceptual understanding in any subject. So, the development of thinking aids learning in areas beyond mathematics.

The development of mathematical thinking is an important skill for the twenty-first century. As quoted by McTighe and Schollenberger (1991), The National Science Board Commission on Pre-College Education in Mathematics, Science, and Technology stated that American educators must return to basics, but the basics of the 21st century are not reading, writing and arithmetic. They include communication and higher problem-solving skills, and scientific and technological literacy—the thinking tools that allow us to understand the technological world around us....Development of students' capacities for problem-solving and critical thinking in all areas of learning is presented as a fundamental goal. (2-3)
The strategy of teaching thinking supports the vision of mathematics education. Teaching thinking also addresses general application of recent learning research and the needs of the future. The importance of skillful thinking beyond the mathematics classroom lends support to the proposed strategy. However, unless the literature on thinking is applicable to the restructuring of mathematics, the strategy is not viable.

Literature on Thinking and Restructuring Mathematics.

Reformers contend that successful reform of mathematics education requires theoretical clarity and relevant examples of practice (Lovitt et al. 1990). As an attempt to clarify the vision of mathematics education and provide a strategy and models of implementation, this thesis adopts this two step approach. The literature on teaching thinking is more extensive in both these respects than the literature on reform in mathematics education.

Availability of role models and curriculum materials. Teaching thinking emerged as an educational priority about a decade before the new vision of mathematics. "Since 1980 especially, skillful thinking has been identified as a priority of instruction in many American schools" (Beyer 1987, 1). In the last decade much was written about teaching thinking. Many individual classroom lessons are available and "educators considering the selection and
installation of one or more of the available cognitive curriculum programs are often confused by the vast array of alternatives" (Costa 1991a, v). Thus, cognitive education is supported by a rich literature of research, theory and practice.

The availability of models aids the teacher attempting restructuring.

Exemplary curriculum materials can help teachers think about their current roles, try out new roles, and modify the way they teach. Models of new instructional approaches are key to change. (Lovitt et al. 1990, 230)

New models are especially important to the mathematics teacher whose background is steeped in the traditional view. By adopting the strategy of focusing on the teaching of thinking, the classroom teacher is assured of models and materials. In contrast, the need for examples of the new vision of mathematics education is unfulfilled.

Not only does the field of teaching thinking provide role and curriculum models, it provides models of the process of restructuring curriculum as discussed next.

Master plans for changing instruction. The literature on teaching thinking addresses the task of altering or replacing existing curriculum. Models are available for gradual implementation in the classroom. "Lesson plan remodeling is a long-term solution that transforms teaching incrementally as the teachers develop and mature in their critical thinking insights and skills" (Paul 1991a, 125). Other models outline steps for planning
thinking skills programs that incorporate several subject areas or different grade levels (Beyer 1988). Thus not only does the field of teaching thinking provide role and curriculum models, it provides models of the process of restructuring curriculum.

The restructuring strategy of teaching thinking addresses the special problem of the experienced teacher, tackles a key target which has importance beyond the subject of mathematics and makes available a literature rich in role and curriculum models. Though it is beyond the scope of this thesis to present a complete review of this literature, the next sections highlight key ideas about thinking and teaching thinking.

**Thinking Skills and Strategies**

In an abbreviated form this section attempts to build a concept of the thinking process and to identify goals for thinking instruction in the mathematics classroom.

**Models of Thinking.**

A first step in restructuring based on teaching thinking is to construct a model of thinking. "Without a common understanding of what we mean by thinking, we cannot even begin...the development of students' higher cognitive performance" (Presseisen 1991, 62). This task is comparable to clarifying the vision of mathematics.
education. However, in accomplishing this goal the experienced teacher is not hindered by decades of counterproductive models and is aided by the current literature on thinking and the teaching of thinking.


E. Paul Torrance identifies creative thinking skills including fluency, originality and elaboration (Swartz 1987). Perkins (1991b) defines creative thinking components and related prescriptions for education as attention to aesthetics, attention to purposes, mobility, working at the edge of one’s competence, objectivity and intrinsic motivation. Costa’s (1991d) model divides thinking skills into input, processing and output phases. Gubbin (1985) synthesizes the ideas of many theorists in a matrix categorizing skills as problem solving, decision making, inferences, divergent thinking skills, evaluative thinking skills and philosophy and reasoning. Marzano et
al. (1991) propose five dimensions of thinking: metacognition, critical and creative thinking, thinking processes, core thinking skills and content area knowledge. This sampling of perspectives on thinking is not presented to confuse, though this may well be a result. The intent is to make the point "that there is no ideal taxonomy because the complex landscape of thinking can be partitioned in many different equally reasonable ways" (Swartz and Perkins 1990, 36). The freedom and an obligation exist for the classroom teacher to select a perspective and goals appropriate to the content, the students, the situation and the teacher. The teaching of thinking is consistent with the call of this thesis for teacher reflection.

It is beyond the scope of this thesis to examine the plethora of models and classifications of thinking. Instead, some guidelines are presented to start the classroom teacher on the journey and to provide context for the curriculum examples in the next chapter.

Guidelines for a Model of Thinking.

Skills and strategies. The literature supports dividing thinking into cognitive skills and strategies. Skills refer to specific, basic capabilities such as comparing, ordering, generalizing, elaborating and visualizing. Strategies refer to complex, multi-skill
processes including conceptualization, decision making and problem solving.

Teachers are advised to choose skills and strategies appropriate to the educational situation. For example, though detecting bias, visualization and patterning play a role in many subject areas, the skills may be selected for special emphasis in social studies, art and mathematics classes respectively.

Given the manifest importance of so many kinds of thinking, it's wisest to accept this as one of those situations where you have to choose on other grounds than 'official best'. Consider your classroom or school system. Ponder what your students might need and enjoy most. Probe the needs of your subject matter and your own interests and enthusiasms. Contemplate what you can handle comfortably. (Swartz and Perkins 1990, 58)

In addition to selecting appropriate skills and strategies, teachers must link essential skills to complex strategies.

It is important to emphasize both specific subskills and their use in decision making and problem solving in some way in teaching thinking. Problem-solving programs alone have as many limitations as teaching thinking skills alone. (Swartz and Perkins 1990, 161)

This can be accomplished by linking specific skills to multi-step strategies in which the skill is used. For example, flexibility, prediction, prioritizing and comparing are all subskills of one decision making strategy (Swartz and Perkins 1990).

As described, models of thinking distinguish between cognitive skills and strategies. In addition, attention is given to another aspect of thinking, the metacognitive.
Metacognition. Defined simply, metacognition is "thinking about thinking" (Costa and Lowery 1989, 64).

Metacognition consists of standing outside of one's head and directing how one is going about executing a thinking task. It involves planning how to carry out the task and carrying it out. It involves, in addition, monitoring one's progress, adjusting one's actions to the plan, and even revising both plan and actions in the process. (Beyer 1987, 192)

Metacognition is dealt with as a special type of thinking. It cannot be classified as a single skill or strategy. Metacognition "is a cross cutting superordinate kind of thinking relevant to all the others" (Swartz and Perkins 1990, 51). It "is an overarching cognitive ability that 'monitors' our other thinking processes" (Costa and Lowery 1989, 65).

Metacognitive capabilities are key to independent thinking. "Well-designed instruction should encourage students to become metacognitive because this puts them in charge of their own instruction" (Swartz and Perkins 1990, 53).

Unless students are helped to become conscious of their own thinking, keep track of what they are doing when they engage in thinking, and assess the effectiveness of what they do, they cannot take control of their own thinking and become self-directed thinkers....Teaching for metacognition helps students become conscious of how they think so that they can control it to increase the efficiency and effectiveness of their thinking. (Beyer 1987, 214-215)

From the perspective of the new vision of mathematics education, "if we want our students to become active learners and doers of mathematics rather than mere knowers of mathematical facts and procedures, we must design our
instruction to help develop their metacognition" (Garofalo 1987, 22).

Metacognition also warrants attention as a source of information for instructional decisions and as an ability associated with achievement. Student metacognitive reports provide a teacher with essential insights. "Teachers' knowledge of children's thinking makes it possible for them to challenge and extend students' thinking and appropriately modify or develop activities for students" (Maher and Davis, 1990, 90). Likewise, research in mathematics classrooms has "found that students' abilities to diagnose and monitor their own understanding is an important predictor of their mathematics achievement" (Peterson 1988, 8).


the major components of metacognition include developing a plan of action, maintaining that plan in mind over a period of time, and then reflecting back on and evaluating the plan upon completion. (Costa and Lowery, 1989, 66)

Swartz and Perkins (1990) base instruction on four increasingly metacognitive levels of thinking: tacit use, aware use, strategic use and reflective use. This sampling demonstrates that, though there is agreement about the importance of metacognition, there are a variety of models
of it. The classroom teacher must select a model of metacognition appropriate to the instructional situation.

**Critical vs. creative thinking.** Thinking may seem to divide into critical thinking and creative thinking. Indeed many books focus on either critical thought (Paul 1990, Halpern 1989) or creativity (Perkins 1981, Davis 1986). If this separation is accepted, it is reasonable for the mathematics teacher to assume critical thinking is the concentration for mathematics instruction.

However, a strict division is not supported by recent theory and research.

Although critical thinking is commonly thought of as evaluative and creative thinking as generative, the two actually complement each other and work together. All good thinking involves both quality assessment and the production of novelty. Critical thinkers generate ways to test assertions; creative thinkers examine the newly generated thoughts to assess their validity and utility. The difference is not of kind but of degree and emphasis. (Marzano et al. 1991, 90)

The separation reflects the outcome of thought more so than the processes involved.

The assumption of mathematics thinking as critical thinking limits the implementation of the vision. Though the assumption is reasonable given the traditional view of mathematics education as a logical sequence resulting in one irrefutable answer, this is not the perception of mathematics espoused by the vision. Creative aspects of
thinking mathematically need to be acknowledged and reinforced.

The moral for educators is to avoid implying that critical thinking and creative thought are opposite ends of a single continuum. This does not mean, however, that specific elements cannot be identified for both critical and creative thought. (Marzano et al. 1991, 90)

It is unrealistic to expect the classroom teacher interested in restructuring mathematics instruction to engage in a comprehensive study of thinking. Fortunately, it is not prerequisite to implementing this strategy for restructuring mathematics instruction. Teachers need to identify skills and strategies appropriate to their situations from a rich literature. The next section describes the identification of the skills and strategies that are the basis of the mathematics curriculum examples to be presented in Chapter IV.

Thinking Skills and Strategies Important in Mathematics.

discuss evaluation of cognitive processes in mathematics (Stenmark 1991).

The implication is an abundance of mathematical thinking skill and strategy lists. There are some lists specific to mathematics. In defining mathematical thinking, the California State Department of Education (1991) "includes analyzing, classifying, planning, comparing, investigating, designing, inferring and deducing, making hypotheses and mathematical models and testing and verifying them" (3). Davidson (1991b) lists aspects of mathematical thinking as including visual thinking, logical reasoning, generalizing, problem solving, patterning, part-to-whole reasoning, whole-to-part reasoning and problem posing. Problem solving is the only thinking process systematically treated to any degree in mathematics curriculum materials.

The implication that lists of mathematical thinking skills and strategies are plentiful is false. The lists cited do not give the impression of being complete nor do the sources indicate more than an intent to illustrate mathematical thinking.

The thinking skills and strategies developed in the curriculum models of the next chapter were the result of teacher reflection and two types of sources. The models of thinking were culled for skills deemed appropriate to mathematics. For example, classification is listed in many models (Presseisen 1991). Secondly, the mathematics
education literature was reexamined for verbs that indicated thinking processes. In the three grade level descriptions of "mathematics as reasoning" (NCTM 1989, 15), recognizing patterns, reasoning spatially, conjecturing, reasoning inductively and deductively and generalizing indicate thinking skills and strategies.

The two skills and one strategy chosen as the basis of the Chapter IV models were judged on a variety of criteria. The skills and strategy selected are recognized as thinking processes in the cognitive education literature. They also are used regularly in a variety of mathematics contexts. The choices represent both skills, "discrete thinking operations" (Beyer 1987, 25), and strategies, "much more complex sequential operations" (25). Furthermore, instruction in these processes incorporates other targets important to the new vision of mathematics education. All have application beyond mathematics. The advice given earlier is heeded; skills and strategies were selected that fit the aims of this thesis.

The two skills selected are classification and pattern recognition; the strategy chosen is conceptualization. In the next chapter instructional methods or techniques are presented that generate student thinking using the two skills and the strategy. An analysis of each skill or strategy is presented with the associated technique.
The skills and strategies listed in this section represent a restricted list. As described in Chapter II, mathematical thinking encompasses many skills and strategies beyond traditional mathematics. As teachers shift instruction to mathematical thinking as in the new vision, students will engage in a range of skills and strategies. The next sections review levels of planning and several guidelines for such instruction in thinking.

Levels of Curriculum Planning for Teaching Thinking

The author of this thesis views the process of planning thinking instruction from three levels. The first is a long-range plan which considers the development of a skill or strategy over a school year or several grades. The second concentrates on a unified segment of instruction, a lesson or possibly a unit, which focuses on a particular skill or strategy. The third level of planning focuses a microscope on one part of a thinking lesson to examine the technique underlying the core thinking activity that engages students in using the particular skill or strategy. The emphasis given to each level in this thesis is determined by the perceived needs of the experienced high school teacher.

Commitment to implementing the vision implies a long-range plan for the development of mathematical thinking. Beyer (1987) views such a plan in six stages: "introduction
...guided practice...independent application...transfer and elaboration...guided practice...[and]...autonomous use" (75). If this thesis addressed system-wide curriculum developers, such models for the first level in the planning of thinking programs would receive more than passing attention. However, this does not fit the reality of the struggling classroom teacher. As the experienced teacher grows into the role prescribed by the new vision, long-range plans may increase in priority. Given that this thesis attempts to build from the reality of the classroom teacher, further discussion of the first level is not included.

This thesis is written for the experienced high school teacher attempting to implement the new vision of mathematics education through the strategy of teaching thinking. As a practical reality, this teacher must first focus on the third level which deals with techniques for thinking activities. From the perspective of an expert on thinking, the third level focus is on teaching for thinking, that is, engaging students in thinking. From the perspective of the mathematics educator, the third level focus is on student-centered tasks that involve students in mathematical thinking. For the experienced teacher this is a sizeable expansion beyond the traditional role and repertoire.

To fully utilize techniques for teaching thinking, the teacher must recognize each technique as the third
level of an inward spiral of curriculum planning. The technique cannot be fully understood, if it is isolated from the entire lesson plan which is the second level. For this reason, the next sections in this chapter present key issues in the teaching of thinking and a lesson plan structure based on these issues. One element of the generic lesson plan is a core activity which engages students in thinking. Specific techniques to engage students in thinking are the focus of Chapter IV. The techniques are better understood if the overall lesson plan is explicit.

Issues in Teaching Thinking

Infused vs. Stand Alone Approaches.

There is a continuing debate on infused versus stand alone instruction in thinking. In the infused approach thinking skills and strategies are taught in standard subject area classes. This involves "infusing teaching for thinking into regular classroom instruction by restructuring the way traditional curriculum materials are used" (Swartz and Perkins 1990, 68). In contrast, the stand alone approach advocates a separate course. The subject matter of thinking is generic and examples cover many fields.

Given that teaching thinking is presented as a strategy for restructuring mathematics instruction, it may
seem that the issue is resolved in favor of infused instruction. However, the debate continues as decisions are made about integrating thinking instruction into existing curricula. Separate units on thinking skills and on problem solving reflect a stand alone approach. Blending instruction in thinking throughout all the topics of the course represents an infused approach.

The infused approach is the choice consistent with the new vision of mathematics education. Mathematical thinking is developed throughout content instruction, not isolated in separate units. Mathematics concepts, real-life applications and problem solving are the context for teaching thinking.

Research on the content specificity of thinking also supports an infused approach. Studies of mathematics learning found

the total number of general cognitive strategies reported by students was negatively related to students' mathematics achievement, but that the total number of specific cognitive strategies reported was positively related to achievement. (Peterson 1988, 13)

Skills as well as knowledge are specialized so "similar thinking skills... take on different character in different subject matter domains" (Prawat 1991, 185).

Occasionally the infused approach is mistakenly linked with indirect, not direct, instruction. However, both direct and indirect methods are used in implementing the vision of mathematics instruction. These methods are reviewed in the next section.
Explicit, Direct and Indirect Instruction.

Need for explicit instruction. Experts in cognitive education (Beyer 1987, Swartz and Perkins 1990, Costa 1991c) advocate explicit instruction. Presentation of thinking skills and strategies is manifest, not implied and precise, not vague or general.

If students are to acquire good thinking skills in the classroom, explicit attention will have to be given to that objective; it is not likely to be realized spontaneously or as an incidental consequence of attempts to accomplish other goals. (Nickerson 1987, 29)

Effective instruction requires clarity about the definition, structure and use of skills and strategies.

Analysis for explicitness. To be explicit about a thinking skill or strategy, one must analyze it completely. Teachers must be clear about "the details of the kind of thinking we want to help students perform...This does not mean just knowing the name of that thinking skill. Rather, it means understanding its deeper structure..." (Swartz and Perkins 1990, 80).

Beyer (1987) details a conceptual model for analyzing a thinking skill or strategy which incorporates definition, procedures, rules and knowledge. Definition identifies the key attributes of the skill or strategy. Procedure outlines the steps and substeps in executing the skill or strategy as well as their sequencing. Rules give various pointers about when to use the skill or strategy, how to
get started, what problems might occur and how to handle them. Knowledge refers to related information. For example, special vocabulary or a list of criteria may be associated with a particular skill. Beyer's model provides a theoretical structure for a complete analysis of any thinking skill or strategy. It is mentioned here as the structure is helpful in organizing the variety of information associated with analyzing a thinking skill or strategy.

A classroom teacher's analysis of a skill or strategy should evolve. Initially, an analysis represents expert ideas culled by a teacher's sense of the instructional situation. In order to start the restructuring process, an analysis of thinking needs to reflect the classroom teacher's situation, not that of a curriculum expert whose primary concern is detailed objectives. With additional research and the experience gained from evaluating initial lessons, modification of the analysis will occur.

The important point here is that the classroom teacher must strive for a clear concept of the thinking identified for instruction. In order to teach a skill or strategy explicitly, the teacher must work from a deep understanding. It may not be appropriate to share all the detail of such an analysis with all students at all stages of instruction, but the teacher's background should provide a clear focus.
A direct/indirect continuum. Direct and indirect instruction form a continuum which varies in the responsibility of teacher and student.

Highly organized forms of direct teaching involve instructors giving students thinking strategies to use explicitly in the form of rules to follow in going through various steps in thinking. Sometimes charts exposing these steps are posted. More indirect methods involve teachers interacting with students in a discussion. The teachers use prompting questions that focus student attention in an orderly fashion. Thereby they structure their thinking. (Swartz and Perkins 1990, 170)

Direct instruction reflects the traditional model emphasizing teacher responsibility. Students receive information structured and presented by the teacher and are responsible for retention. Indirect instruction shifts responsibility for identifying and analyzing information to the students. The teacher does not abdicate responsibility, but rather shifts it to guiding students. The use of the term 'direct instruction' causes difficulty. Some use it synonymously with explicit instruction. Others use it to mean instruction in which the teacher defines and analyzes a thinking skill or strategy with little or no student input. In this thesis, direct instruction will refer to the latter.

Use of direct and indirect methods. Both direct instruction and indirect instruction are utilized in a continuum in any part of a thinking lesson. In particular, effective explicit instruction incorporates techniques that fall throughout the direct-indirect continuum. There are
times when it is appropriate to introduce a thinking skill with a teacher lecture defining, analyzing and modeling the skill objective. In other instances, students engage in the skill and then define and analyze it with minimal guidance from the teacher. In teaching thinking the important difference is not between direct and indirect methods, but between approaches that explicitly teach thinking and those that never address the definition, components or steps associated with a thinking skill or strategy.

**Necessary, but not sufficient.** Explicit instruction, through direct and indirect methods, is needed for effective teaching of mathematical thinking, but it is not enough. Explicit instruction "might be the most effective method for promoting students' achievement of lower-level skills in mathematics, it may be necessary but insufficient for enhancing achievement of higher-level skills" (Peterson 1988, 5). Thinking skills and strategies are not acquired by passive listening, but rather by active involvement. In addition to analyzing the targeted skill or strategy, teachers must involve students in thinking.

**Engaging Students in Thinking.**

Active engagement in thinking promotes student's understanding and ability to use the desired skill or strategy. Current learning research supports the
constructivist view recommending "activities that allow students to process the content actively and 'make it their own'" (Brophy 1992, 5). The emphasis is on understanding and application in various situations rather than the automaticity of lower-order skills.

Students test and refine their thinking through small group activities. Such activities serve as a useful bridge between whole-class activities where the teacher is in charge, and solo activities where students are on their own. Indeed, we consider it likely that small group activities may be the single most useful mode of interaction, the one that should occupy the highest percentage of activity time, in the classroom as students learn to think. (Swartz and Perkins 1990, 32)

Though teachers may initially guide students, instruction quickly moves to interaction among peers or autonomous thinking.

When students are actively involved, the responsibility to think lies with the student, not the teacher.

Higher-order thinking may require a less direct instructional approach that transfers some of the burden for teaching and learning from the teacher to the student and promotes greater student autonomy and independence in the teaching-learning process. (Peterson 1988, 5)

This is the key to shifting the role of the teacher from the traditional model to the new vision of mathematics education. Thinking and learning become student-centered and the teacher's role is that of coach and resource. Techniques to accomplish these goals are the focus of Chapter IV.
Effective instruction in thinking must provide activities that generate thinking. Specifically, teachers must develop techniques to stimulate student thought. However, the success of these activities are influenced by student disposition.

Thinking and Affect.

"Researchers have recently emphasized the influence of the more affective aspects of thinking on students' cognitive performance" (Presseisen 1991, 61). Paul (1991b) proposes a taxonomy classifying thinking as cognitive and affective strategies. Ennis' (1991) view of critical thinking curriculum divides goals into dispositions and abilities. Costa (1991b) includes as goals of intelligent behavior persistence, decreasing impulsivity, striving for accuracy and precision, a sense of humor and risk taking. Marzano et al. (1991) state that the elements associated with both critical and creative thinking can be divided into skills and dispositions. "These additional elements of thinking represent an affective dimension of thinking. They support and drive thinking" (Beyer 1987, 213).

So the classroom teacher must attend to the affective aspects of teaching thinking. However, unfortunately for the high school teacher, the research also reports that "attitudes, values and dispositions are formed early in life. Indeed, in most cases they are established rather
firmly by the time youngsters enter their junior high school years" (Beyer 1987, 21). Classroom teachers must teach thinking, even though high school students have not developed appropriate attitudes and perceptions. The dispositions that support thinking should be explicitly presented as the students are engaged in activities to foster them. At minimum, teachers should share the purpose of instruction with students so that they are aware of the intended benefit and value. All instruction should attempt to motivate and focus students' interest and attention.

Modeling and insisting on student behaviors that illustrate these dispositions will help, too. Developing dispositions supportive of effective thinking clearly is a challenge requiring a continuous and long range effort. (Beyer 1987, 21)

Another aspect of thinking that requires repeated attention over time is transfer. Like disposition and the structure of a thinking skill or strategy, the attention required is explicit.

**Transfer.**

It is a basic assumption of American education that knowledge and skills acquired in school will be used outside the classroom. This is the concept "of transfer--something learned in one context helps in another. Transfer goes beyond ordinary learning in that the skill or knowledge in question has to travel to a new context" (Perkins and Salomon 1991, 215).
Transfer is a continuing concern of those in the field of thinking education.

Research has tended to show that, while some approaches to teaching thinking may stimulate good thinking in the classroom, there is little or no transfer to learning in other classes or to everyday thinking patterns. Students simply do not do the same sort of thinking outside of the particular contexts of instruction, even in situations which clearly call for it. (Swartz and Perkins 1990, 174)

Thus, the context of learning limits transfer outside the classroom.

A theoretical distinction between "low-road and high-road transfer" (Perkins 1987, 51) helps reconcile the disparity between the educational assumption and the research findings. In low-road transfer the learning and transfer contexts are superficially similar. The knowledge or skill is routine or automatic. Low-road transfer seems to occur without attention. For example, "interpreting a bar graph in economics automatically musters bar graph interpretation skills acquired in math" (Perkins and Salomon 1991, 218). In contrast high-road transfer is not likely to occur without explicit attention.

High-road transfer occurs by way of mindful abstraction from the context of learning and application to another context. It demands the conscious effort of the learner in seeking generalizations and applications beyond the obvious.... (Perkins 1987, 51)

Therefore, to be effective, instruction in thinking must attend to high-road transfer.

Transfer requires repeated, varied practice.

Introduction of a skill or strategy is followed by practice
in similar contexts. This "hugging" (Perkins and Salomon 1991, 220) technique limits the context in which a thinking skill or strategy is executed. The ease of such low-road transfer allows students to develop proficiency in the skill or strategy. However, the goal is to gradually broaden the context to promote high-road transfer. The technique of "bridging" (Perkins and Salomon 1991, 220) is designed to abstract elements of a thinking skill and identify connections with other contexts. These are criteria of high-road transfer. Teaching for transfer will vary as student familiarity with the thinking skill or strategy develops.

Infused instruction, explicit analysis and presentation, attention to affect and concern for transfer are aspects of teaching thinking that are reflected in effective instruction. The next section incorporates these goals into a structure for a lesson on thinking.

A Lesson Plan for Cognitive Instruction

Elements of a Thinking Lesson.

Earlier sections highlight important aspects of thinking and key issues related to teaching thinking. The focus of the thesis narrows to the second level of curriculum planning at this point, as theory and research are incorporated into elements of an effective thinking lesson. Five elements are identified: focusing, explicit
skill/strategy analysis, core thinking activity, metacognition and transfer.

Focusing: Attention to Intention.

The focusing element of a lesson on thinking attends to student disposition. As discussed earlier, affect effects thinking. This element of a lesson is designed to inform students of the purpose of instruction and to motivate them to acquire the skill or strategy.

The key function of the focusing part of a thinking lesson is to clarify purpose. Teachers must recognize "the importance of gaining student's acceptance of what is going to take place: what the mathematics lesson is about, what they will get out of it" (Lovitt et al. 1990, 233) When students view the skill or strategy as worthwhile, they will invest effort in its attainment. Such a disposition needs to be nurtured over time.

Ideally the focusing element motivates the student and displays the targeted skill. For example, an intriguing puzzle piques students' interest and elicits a particular thinking strategy in its solution. Cartoons and posters can catch student attention while introducing a thinking skill. Well conceived anecdotes help students shift gears to the learning at hand.

Usually the focusing element is presented first in a lesson plan sequence. In introducing a thinking skill or strategy, it is logical to build a positive disposition
before instruction. This is especially true when students have not had much exposure to thinking instruction. The best conceived lessons may be dismissed when students do not understand their purpose or relation to mathematics (Borasi 1990).

Teachers can employ a range of direct and indirect instruction in planning the focusing element. When students (or the teacher) are being weaned from the traditional model or when a teacher is concerned about conserving time, a lecture format can be utilized to identify a particular thinking skill and to expound its importance. In contrast, a teacher may ask students to solve a puzzle, then challenge the students to identify a thinking skill used. Students can be asked to generate rationales as well.

A variety of possibilities exist for the focusing element. However, the key is to design the lesson to put students in a frame of mind to attend to the thinking at hand.

Thinking Analysis: Explicit Instruction.

Earlier it was stressed that the effective teacher analyzes the targeted thinking and provides explicit instruction based on this analysis. Though it may be inappropriate to present the complete analysis, there should be explicit instruction beyond a label and simple definition. Related vocabulary and concepts should be
taught. These should include guidelines for application and execution of the skill. Students should taste the knowledge base behind thinking.

A concrete representation helps make the skill or strategy analysis explicit. A representation might take the form of a list of "attention points whose spirit is to flag important things to attend to in doing this kind of thinking, no matter what their order is" (Swartz and Perkins 1990, 145). A variation is a procedural list stating a sequence of steps for executing a particular thinking strategy. Other possibilities include graphic organizers such as "Venn diagrams . . . concept maps . . . causal chain maps . . . and . . . thinking wheels" (Clarke 1991, 226-227).

Modeling often aids in making aspects of thinking explicit. Teachers may introduce a skill, then demonstrate its use. Students may share effective use of a skill with each other. In either case, it is important to link the model with the verbal analysis.

As students become more familiar with the skill or strategy the initial analysis should be elaborated. This may be done directly by the teacher or indirectly by incorporating students' reflections. As a skill becomes ingrained, conscious awareness of its structure fades. However, before this happens students need repeated experience putting the skill or strategy analysis into practice. This leads to the core thinking activity in every lesson.
Core Thinking Activity: Technique to Generate Thinking.

There is one important feature of instructional design, however, that cuts across these diverse approaches....Quite simply, it involves getting students to use the sorts of thinking the program is concerned to help students improve. Although this may seem obvious, the importance of actively engaging students in the thinking we are trying to teach them is something that has been stressed by most writers who have promoted teaching thinking. (Swartz and Perkins 1990, 166)

This thesis reinforces the importance of opportunities for students to experience and practice the desired thinking. The core thinking activity component of a thinking lesson engages students in thinking.

Core thinking activities are structured to guide students to use the targeted skill or strategy. The basis of this structuring is the analysis of the skill or strategy done by the teacher. If the analysis of a thinking strategy involves executing a multi-step procedure, then core thinking activities designed to teach the strategy must allow students to experience all steps. If application of a skill involves selecting from a number of alternatives, then tasks should incorporate a variety of choices.

Structured thinking activities are not dependent on direct teacher leadership. On the contrary, activities that keep the teacher in the role of coach or resource maximize student participation and free teachers from the traditional role. Though it is challenging to design activities that allow student independence while
structuring the desired thinking, this is key in teaching thinking and implementing the new vision of mathematics education.

Since implementation of the vision depends on a shift from traditional roles, this is the component of a thinking lesson put under the microscope in the next chapter. Techniques that are the basis of core thinking activities are presented and analyzed.

The application of techniques to generate thinking about the material of mathematics is the core of restructuring mathematics education.

There are two key ingredients in successful efforts to engage students in active thinking:
1. the accessibility of adequate information students can use in their thinking;
2. the use of instructional techniques to prompt the sort of active thinking about this information that is the goal of the lesson.

(Swartz and Perkins 1990, 167)

Mathematics is the information. Techniques that generate thinking skills and strategies are the instructional prompts.

Examples of instructional techniques to stimulate student thinking are presented in Chapter IV. However, there are two more components of an effective thinking lesson to review before moving to this level of planning.

Metacognition: Regulating and Planning.

Effective thinking lessons incorporate some degree of metacognition. "Because metacognition is so important to
skillful thinking, most experts agree that any serious
effort at teaching thinking skills must also help students
develop their skills of thinking about thinking" (Beyer
1987, 192).

There are a variety of ways to incorporate
metacognition into a lesson on thinking. Students often
retrospectively review their thinking processes, but a
lesson may require students to engage in metacognition as
they think. Problem solving research (Schoenfeld 1982)
sometimes adopts a think aloud protocol in which throughout
the solution of a problem subjects talk about what they are
thinking. A classroom variation is the paired problem
solving method which "involves two people working together
on problems, with each person having a specific role as
problem solver or listener" (Whimby and Lochhead 1984, 2). The benefits of this procedure include increased awareness
and comparison with others' methods.

Metacognitive reflection may be shared in a group or
recorded privately. Whole class discussions are efficient
ways to summarize key aspects of a thinking skill or
strategy. Students can benefit from a comparison of
methods. One method may be judged more effective in the
given situation or students may conclude several methods
are equally productive. Journals are recommended for
private metacognition. Unlike many group discussion,
journal writing guarantees a student the opportunity to
participate in metacognition.
The record also provides an opportunity to revisit initial perceptions—to compare the changes in those perceptions with the addition of more data; to chart the processes of strategic thinking and decision making; to identify the blind alleys and pathways taken.... (Costa and Lowery 1989, 72)

Metacognitive reflection may be highly structured or open-ended. Teachers may be very directive in controlling a whole class discussion or students may complete a worksheet with a sequenced list of questions that directs analysis of their thinking. An open-ended approach might simply ask students to react to how they thought through a particular task. Sentence completions like 'Today I learned...' or 'The hardest part was...’ may be used to spur metacognitive thought.

In analyzing approaches to metacognitive instruction, Swartz and Perkins (1990) discern three approaches. They identify

- aware uses of thinking skills (using thinking terms)
- strategic uses of thinking skills (providing a list of components or a series of steps)
- reflective uses of thinking skills (helping students monitor their thinking by describing it, helping students reflect on effective ways of doing this type of thinking, and then asking them to direct their thinking accordingly). (187)

Thinking about thinking in the sense of aware and strategic uses is built into other parts of the effective lesson plan. High school teachers should aim for reflective uses. Their students are capable of metacognitive reflection to regulate and prescribe thinking processes. A shift from awareness to reflection is also a goal of transfer which is the fifth element of thinking lessons.
What can one do to maximize transfer of training for thinking...? I believe this to be the fundamental question in the teaching of thinking skills. Without far-spreading transfer of training, instruction in thinking skills is not terribly meaningful. And psychological research has shown that transfer of training does not come easily. One must teach for transfer, rather than merely hoping or even praying that it will occur. (Sternberg 1987, 258)

A first step in teaching for transfer is to increase students' awareness of the challenge. Teachers can point out to students how a skill might be used in a different context and how previously learned skills might be applied to a new situation. As students' awareness increases, the responsibility for identifying opportunities for transfer should shift to students. Whether in a brief exposition or as the focus of extended analysis, the question of where else this might be used should be addressed for every skill or strategy taught.

Transfer is also enhanced by crafting varied practice into lessons on thinking. As a skill develops, the practice context should vary. This requires planning that looks backward to consider earlier lesson contexts.

Teachers can promote high-road transfer to dissimilar situations by anticipation.

Reflective and deliberate practice based on a blending of a metacognitive awareness of the appropriate forms of thinking to be used and reflection on new and varied examples is well-researched as an extremely effective classroom strategy in teaching thinking. (Swartz and Perkins 1990, 85)
For example, teachers and students can analyze the general conditions leading to the use of a particular skill, then brainstorm other situations in which the underlying conditions are similar despite surface differences.

Here transfer is discussed as a lesson component, yet in reality it cycles back to the beginning of another lesson. As with skill/strategy analysis and metacognition, transfer must be developed over time.

Application of the Lesson Plan Structure.

The lesson plan structure for teaching thinking includes focusing, explicit analysis, a core thinking activity, metacognition and transfer. Though each component is discussed in the order listed, this sequencing is only one possibility. For example, the skill/strategy analysis and the core activity might be reversed in order to enhance student input in elaborating the analysis.

Each lesson plan component is presented and discussed separately. However, the blending of elements is appropriate. Solicitation of student ideas for transfer might be a segment of a metacognitive discussion or metacognitive reflection on the previous use of a particular strategy might serve as the focusing element of a lesson. Once again teachers must reflect on how their situation is best served.

In this part of the chapter, general guidelines for a model of thinking and issues related to teaching thinking
have been distilled into a plan for action. This lesson plan structures implementation of teaching thinking. The processes of thinking become the content of the lesson. The traditional content of mathematics becomes the material on which the thinking process is practiced. This combination and realignment of thinking processes and mathematics is the restructuring strategy recommended for high school teachers struggling to implement the vision of mathematics education. This emphasis on teaching thinking does not diminish the goals of the new vision. Rather, as described in the next section, it supports the other targets of the vision.

**Mathematical Thinking and the Other Targets**

This section returns to teaching thinking as the strategy for implementing the new vision of mathematics education. To further support this restructuring strategy, aspects of teaching thinking are summarized in terms of the targets or goal categories identified in Chapter II. Teaching thinking clearly enhances deep understanding of concepts and schemas, mathematical thinking, communication about mathematics, a positive disposition toward mathematics, student-centered tasks, a variety of work formats, mathematical tools and assessment alternatives.
A focus on thinking strengthens the skills and strategies leading to a deep understanding of concepts and schemas.

There is no way around the need of minds to think their way to knowledge. Thought is the key to knowledge. Knowledge is discovered by thinking, analyzed by thinking, and, most importantly, acquired by thinking. (Paul 1990, xv)

Thinking is the means to the end of constructing mathematical knowledge.

Instruction in thinking incorporates different modes of communication. Spoken and written dialogue is used; internal, one-way and two-way conversations are incorporated. All are a part of communicating about mathematics in the new vision.

Dispositions supportive of effective thinking aid in learning mathematics. Thinking dispositions include "independence of mind...intellectual curiosity...intellectual courage...intellectual perseverance...and faith in reason" (Paul 1991b, 78-79). Similarly, the new vision identifies self-confidence, a positive attitude toward learning and perseverance as goals of mathematics instruction.

The criteria of student-centered tasks are applicable to core thinking activities. In both, active student involvement is the key ingredient. By participating in the core thinking activity students "actively use the thinking skill that is the target of the lesson" (Swartz and Perkins 1990, 74). Multiple strategies, a rich context and teacher
as coach are other common elements. Core thinking activities are student-centered tasks.

Instruction in thinking utilizes individual, small group and whole class work formats. However, cooperative learning formats are stressed. "We need others...to probe and question our thinking, to present their thinking as a contrast that enlivens and stimulates ours" (Paul 1990, xv). Thinking instruction matches the new vision in emphasizing collaboration while incorporating a variety of work formats.

Cognitive instruction utilizes literal and figurative tools as does the new vision of mathematics education. Though beyond the scope of this thesis, some thinking skill programs employ computer technology (Pogrow 1991, Meeker 1991, Educational Testing Service 1991). In terms of this thesis, concrete tools like computers, calculators and mathematics manipulatives can be regarded as means of engaging student interest. "Theory-embedded tools,... tangible teaching/learning devices that are material embodiments of theoretically valid teaching/learning ideas" (McTighe and Lyman 1991, 243), are figurative tools clearly appropriate to thinking and mathematics. Examples include heuristics, procedural lists and graphic organizers.

Since advocates of the new vision regard assessment as determining the current state of knowledge, assessment is on-going. Proponents of thinking instruction (Beyer 1987, Costa and Lowery 1989, Swartz and Perkins 1990) also
imply on-going assessment in a continuing spiral of stating, experiencing, assessing and refining thinking skills and strategies. For both groups self assessment and peer reaction supplement traditional teacher evaluation.

The main purpose of this chapter is to explain and justify the strategy of restructuring mathematics education through teaching thinking. In accomplishing this, it emerges that the goals and tactics of the new vision and of cognitive science overlap and intertwine.

**Knowledge as Design**

*Knowledge as Design*

**A Theory across the Turning Point.**

This thesis presents the classroom teacher with a two step approach to attaining the new vision of mathematics: clarify the vision as a framework of instructional targets and implement restructuring by teaching thinking. Chapter II presented a foundation for the vision grounded in the mathematics education literature. This chapter employs the cognitive education literature in a rationale for restructuring by teaching thinking. Though guidelines and examples have been sketched, specifics of implementing curriculum models are reserved for the next chapter. In Chapter IV the thesis turns to the details and practicalities of implementation.

But before the shift of emphasis from the theoretical, one more theory needs to be examined. The
theory of "knowledge as design" (Perkins 1986, 1) provides a bridge between the synthesis of information on mathematics and cognitive education and the generation of implementation models. The idea of knowledge as design is reflected in the framework of targets, the lesson plan for thinking and the techniques for implementation.

The Theory.

Perkins (1986) rejects the concept of knowledge as information and develops the idea of knowledge as design. This perspective is applied to everything from common objects, such as a thumbtack, to mathematical abstractions, such as the Pythagorean theorem. In this thesis, the theory underlies the synthesis of information and the generation of instructional techniques.

Knowledge as design is based on the concept of a design and four design questions. To define the concept of a design, "one might say that a design is a structure adapted to a purpose" (Perkins 1986, 2). A design is a human construction. Knowledge as design implies "application and justification that make it meaningful" (Perkins 1986, 4).

This emphasis on purpose, structure, application (models) and justification (arguments) is reflected in the four design questions.

To put it succinctly, virtually any product of human effort, including knowledge, can be understood better with the help of four design questions: What is the purpose? What is the structure? What are some model
cases (concrete examples that bring the matter in question closer to perceptual experience)? What are the arguments for or against the design? (Perkins 1991a, 295)

Purpose defines the function and importance of a design. Structure refers to the key components, parts, properties or steps that organize a design. Models are concrete representations of the design that may take the form of physical objects, verbal descriptions or demonstrated behavior. Arguments are typed as explanatory, evaluative, justificatory, hypothetical and persuasive (Perkins 1986). These questions are the basis for learning, creating or teaching a design.

If the design questions are examined and connected to one another, knowledge is no longer inert or disconnected information. These questions lead to deep understanding. Knowledge as design then becomes a powerful metaphor for learning and teaching (Perkins 1986). Students (or teachers) striving for understanding of a simple phenomenon or of abstract principles can organize study around these questions. Teachers attempting to facilitate learning should incorporate these four questions as aspects of instruction. As described below the questions are applied in the learning and teaching of this thesis.

Application in this Thesis.

The framework of instructional targets presented in Chapter II is an example of knowledge as design. The
purpose of the framework is to clarify the vision of mathematics education. The eight targets are the components of the structure. Each target is illustrated with options or models. The target rationales are explanatory arguments and the discussions of variations, modifications and qualifications are evaluative arguments.

The elements of the thinking lesson plan is a second example of knowledge as design. The purpose of the lesson plan is to create an effective unit of thinking instruction. The guidelines for teaching thinking are arguments which identify what is necessary for effective thinking skills instruction. The structure comprises the five elements of focusing, skill/strategy analysis, core thinking activity, metacognition and transfer. Models are sketched in the suggestions given.

In these examples, Perkins' knowledge as design concept serves as a metaphor for learning. Knowledge as design helps to structure the existing information from the literature of mathematics and from the literature of cognitive education. As the page is turned to Chapter IV and the details of implementation are unfolded, knowledge as design is used as a metaphor for instruction. As the microscope zooms in on techniques for core thinking activities, knowledge as design is the basis for the generation and presentation of ideas.
CHAPTER IV
TECHNIQUES FOR TEACHING THINKING

Overview

This chapter focuses on three techniques that combine the framework of targets and the core thinking activity of a cognitive skills lesson. The techniques are designed to stimulate mathematical thinking through student-centered tasks. Each serves as the basis of the core thinking activity in a lesson to teach thinking.

The chapter begins with an explanation of the emphasis given to pedagogical reasoning in presenting the techniques. This often neglected aspect of modeling instruction is viewed as a key to success. In the next section, knowledge as design is revisited as the format for introducing the techniques. Purpose, model, structure and arguments as applied to the techniques are explained.

Three techniques for generating thinking are presented: the card sort technique, the equation/graph pattern technique and the concept explorations technique. Students classify information using the card sort technique. Equations and graphs are examined for relationships in the equation/graph pattern technique. The concept explorations technique allows students to explore a concept through various mathematical diversions. The emphasis on pedagogical reasoning is reflected in the
discussion of decisions and modifications made by teachers when using the techniques. The techniques were designed to stimulate mathematical thinking through a student-centered task. A brief analysis of the relationship to targets other than mathematical thinking concludes the discussion of each technique.

**Pedagogical Reasoning**

The new vision of mathematics education is synthesized and a strategy for implementing the vision is proposed. This chapter moves to the practicalities of restructuring by providing examples of implementation. However, the techniques elaborated are wrapped in pedagogical reasoning—a key to successful change.

"Pedagogical reasoning is the 'intellectualization' or deep thinking of what good teachers do and why they do it. It might well be called the 'wisdom of practice'" (Lovitt et al. 1990, 232). The techniques presented here are modeled, but emphasis is given to analyzing the components of instructional decisions. An attempt is made to expose the structure of techniques that generate student thinking, so that experienced teachers can adapt them for their own classrooms. As additional techniques are discussed, less space is given to describing classroom action and more is invested in reflection on the technique. In an attempt to expand the experienced teacher’s
repertoire, this thesis depends on the teacher’s knowledge and understanding of classroom reality.

Format for Presentation of Techniques

The format for presenting the techniques is based on Perkins’ (1986) theory of knowledge as design. The discussion of each technique includes the purpose, a model, the structure (instructional components) and explanatory and evaluative arguments (reflections, variations and vision implementation).

Purpose.

In the section on purpose, the technique is outlined, the key thinking skill or strategy is described and a brief rationale for teaching the skill in the mathematics classroom is stated. As these are techniques for teaching thinking, the purpose centers on a cognitive skill or strategy, not on a content objective. However, each technique can be used with a variety of content topics. Though each technique targets one skill or strategy, in reality all techniques stimulate several.

Model.

Each technique is modeled for a first year algebra class. The format used to present each model varies. Each incorporates the elements of an effective thinking lesson.
as presented in Chapter III. The technique itself is the basis of the core thinking activity. The lesson plan elements of focusing, skill/strategy analysis and metacognition are explained as the setting for the exercise of the technique. Awareness of high-road transfer and low-road transfer is found in the models. However, opportunities for high-road transfer are not implied until the sections suggesting various additional uses of the technique.

It must be emphasized that the technique and its analysis is the unique contribution of this chapter. Techniques that generate student thinking are the focus, but the technique is only part of the lesson modeled. To illustrate the use of the technique, a lesson is described, but the exercise of the technique, not the entire lesson, is the concern. The technique is the knowledge whose design is analyzed. This is important to remember as the discussion shifts from the model to the structure of the technique.

Structure.

The instructional components of the technique are listed and briefly described. The structure revealed in an analysis is of the instructional decisions the teacher needs to make to implement the technique. The components are not procedural. This focus on underlying instructional
decisions is elaborated in the further discussion of the technique.

Arguments.

The sections on reflections, variations and the relation to other framework targets present explanatory and evaluative arguments. Some comments share the reasoning behind the selection of certain alternatives. Possible pitfalls are highlighted. Possible variations are mixed with comments about the technique. The technique is examined for relationships to all targets of the vision framework.

The goal of this chapter is to stimulate teacher reflection, modification and thus ownership of the techniques. To this end, the techniques are suggested expansions of an experienced teacher's repertoire. The teacher's expertise is necessary, as successful use of the technique in a particular setting relies on refinements made by the teacher. It is stressed that the emphasis of this thesis is to present techniques in terms of their instructional components, not in terms of procedures applied in a particular course.

The Card Sort Technique

The card sort technique is designed to engage students in the thinking skill of classifying. Cards with
Purpose of the Technique.

The purpose of the card sort technique is to immerse students in the thinking skill of classifying. In using the technique, students first examine information on cards for similarities and differences. Next, categories for the information are formulated and labeled. Students then sort the information into the categories chosen. In actual use, students cycle repeatedly through these three steps.

The cognitive skill of classifying incorporates comparing, contrasting and categorizing. Similarities and differences are examined in the process of identifying the commonality that puts data into the same class or category. Categorizing refers to placing information in predefined categories. Therefore, classifying subsumes categorizing. When classifying, the categories must be defined as well.

Classifying is important for mathematics students. Classification of patterns is key in the study of mathematics. Development of the skill aids in recognizing the multitude of classification systems that form the
structure of the discipline. For example, students must understand the classification of the various sets of numbers and their associated properties to successfully complete Algebra I. Also, the cognitive skill of classifying is an effective learning tool. In the process of creating a classification system, students increase deep understanding of concepts. For example, students who have not classified algebraic manipulations often are unable to link given exercises with the appropriate operation on a chapter test. This happens even though the student did well on homework and quizzes covering one or two chapter sections.

A Model Lesson Incorporating the Card Sort Technique.

This lesson uses the card sort technique during the opening days of school in an Algebra I course. The lesson reviews pre-algebra topics from the previous year and sets the tone for active student involvement in learning. Cards contain pre-algebra concepts, properties and key vocabulary. Students classify the cards into topics. As a class, the group outlines are discussed and a list of guidelines for classifying is generated.

The model lesson is presented as a formal lesson plan. Step by step specifics are given. Detail extends to possible phrasing of teacher directions and discussion questions.
LESSON TITLE:

Pre-algebra Review Sort

COURSE:

Algebra I, college-bound freshmen

TIME:

Approximately one and a quarter hours spread over three periods before textbooks are distributed

GOALS:

Knowledge: To review topics in pre-algebra
Cognitive Skill: To develop a classification system
Social Skill: To contribute ideas in group

MATERIALS:

1. Transparency with following list:
   basketball  track  cross country
   swimming  football  field hockey
   soccer  tennis  swimming
   softball  baseball  hockey

2. A set of index cards for each student. The cards measure 3 x 2 1/2 inches or half a 3 x 5 index card. Each contains one word or phrase from the table of contents of the textbook used the previous year.

INTRODUCTION (DAY 1, LAST 15 MINUTES OF CLASS):

1. Introduce activity. "This year you will have many opportunities to discover and rediscover mathematics for yourself. In the process, you will develop your mathematical thinking skills.

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Tomorrow we will be using a key thinking skill in reviewing key pre-algebra topics. As a quick demonstration, try the following.

2. Show list of words on transparency. "How would you put these words into groups? Why?" List suggestions on board as students respond.

3. Give prompts, if needed: "Is there another way?" "Is there a way to put them in three groups?"

4. State purpose of activity. "To group these sports, you must identify a similarity that groups several sports in the same category."

"Classifying is an important tool for organizing knowledge. In the course of everyday living you have already done a lot of classifying. Think of the categories we use for food: fruits, vegetables, meat and dairy products."

"Organizing your knowledge of mathematics will help you know when to apply the facts and skills you have learned. If we are aware of and refine our knowledge arrangement, we will better understand and remember what we have studied."

"Tomorrow's activity will review math topics covered last year, while making you more aware of your organization of mathematics knowledge."
HOMEWORK (DAY ONE):

1. Pass one packet of cards to each student. Each card has a term or phrase indicating a pre-algebra topic, such as 'addition of integers', 'order of operations' and 'exponents'.

2. Give assignment. "Tonight go through these cards and identify any of the words or phrases of which you are unsure. Be prepared to ask about meanings at the start of class tomorrow. Consider your priorities. Are there any questions about the assignment?"

ACTIVITY (DAY TWO, HALF AN HOUR):

1. Divide students into groups of three or four based on seating arrangement.

2. Instruct students that their task has three parts. As each part is described, write the underlined words on the board.
   a. "Clarify any meanings of the information on the cards. Use me as a source of last resort."
   b. "Classify the cards into topics based on the mathematics you have studied already. Think of creating chapter titles for a pre-algebra textbook. This is the heart of the task."
   c. "Finally each group needs to keep a record of their classification system to share with others tomorrow."
3. Answer questions. Encourage students to experiment with categories by physically sorting cards in different arrangements.

4. Monitor time. Facilitate only as needed.

5. At end of group work, collect group record for class discussion.

HOMEWORK (DAY TWO):
Assign a metacognitive journal entry. "Tonight, think about this activity for five minutes, then write a paragraph about what you have learned. There is no right answer. Your thoughts may center on the mathematics of pre-algebra, the experience of working in a small group or something else."

SUMMARY AND METACOGNITION (DAY 3, HALF AN HOUR):
1. Ask groups to share the results of their classifying. Cycle through the following questions as the groups report. Use the questions only if students need prompts. Record category names on board.
   a. "What categories did you identify?" List categories on board. Ask for clarification of the labels as appropriate.
   b. "What criteria put a phrase in this category?"
   c. "What are some of the phrases you put in this category?"

2. Define classifying as "arranging into groups on the basis of (a) shared or common
characteristic(s) or attribute(s)” (Beyer 1988, 327).

3. Shift focus to generating a list of "Guidelines for Classifying". Record the list on a transparency. Question to incorporate "How to do it?...[and]...What to do if..." (Beyer 1988, 327) points about the skill of classifying.

a. "What did your group do to classify this list."
b. "What was helpful in creating these categories? What suggestions would you make to the next group that has to do this activity?"
c. "Did you come up with the category labels all at once?"
d. "Did you change the category labels as you worked?"
e. "Did you have any phrases that didn’t fit? What did you do with phrases that didn’t belong in any category or belonged in more than one category?"
f. "Why do you think groups came up with similar categories?"
g. "Why do you think groups came up with different categories? What did your group assume was the basis for the categories?"
TRANSFER AND HOMEWORK (DAY THREE):

"Tonight’s homework asks you to think about this activity. It has three parts and each should be done as a journal entry."

1. "Think about your knowledge of pre-algebra. Do you feel comfortable picking up where your mathematics course last year ended? If not, what is most important for you to review?"

2. "Think about the various ways of classifying the information on the cards. Describe one modification you might make in your group’s classification system as a result of today’s discussion."

3. "We will use this technique of sorting information cards again. For example, to review for the midyear exam, you will classify problems from practice tests. Tonight, reflect on advice you would give yourself when you classify information cards later in the year. In your journal give at least three suggestions."

Shift of Focus from the Lesson to the Technique.

At this point, it is important for the reader to shift focus with the text of this thesis. In the last section, the model lesson using the card sort technique was described. Now, the lens of analysis zooms in on the core thinking activity used as part of the lesson. In the next
two sections, the instructional components of the technique used in the activity are identified and discussed. The purpose of the discussion is to increase the likelihood of teachers tailoring the technique to their classrooms.

**Instructional Components of the Technique.**

The card sort technique which engages students in classifying is the basis of the core thinking activity. The technique has five instructional components:

1. purpose of sort,
2. information cards,
3. work format,
4. report of classification system and
5. refinement of system.

The purpose or goal of sorting information is identified by the teacher. In the model lesson, the aim is to classify the given information into review topics. Information cards contain a word, phrase or piece of data to be included in the classification system. The teacher selects the most appropriate work format: individuals, pairs, small groups or whole class. In the model, the technique tasks are accomplished in a combination of individual and small group formats. The report mandates commitment to a classification system. Once the classification is complete, the system created or the original information can be refined based on reflection or new information.
Reflection on and Variation of Components.

The teacher must decide how to use the card sort technique in a particular class situation. To do so, each instructional component is examined. Thoughts and possibilities are shared below, but the intent is to stimulate the teacher's own ideas.

**Purpose of the sort.** A distinction must be made between the overall purpose of the card sort technique and this component, that is, the purpose of the particular sort done by students. The overall purpose of the technique is to engage students in classifying. In the model application of the technique, the purpose of the sort is to classify pre-algebra topics for review.

The purpose of the sort may vary from open-ended to structured. For example, in the model lesson the purpose is to sort review topics. In the version presented, the students select categories based on their analysis of the information provided. The approach is more directed if the teacher gives examples of or suggestions for categories. In some situations, it might be judged necessary to give the students a predetermined set of categories. However, if this is done, the skill students practice is categorizing not classifying.

In the model the teacher's oral directions inform students that review topics are to be the basis of the classification system they design. A more open-ended
approach would be to state the purpose as classifying for review without the emphasis on topics. The result might be classification systems based on level of difficulty or degree of mastery rather than content topic. Most challenging is for cards to be prepared without assumptions about the categories to be created. For example, students can classify the problems from recent final examinations.

Information cards. Each information card contains one piece of data. In the model lesson, the cards contain words and phrases related to the previous year’s work. The table of contents of a pre-algebra text serves as source.

A wide variety of information can be used with the card sort technique. Concepts, properties, operations, theorems, procedures, exercises, word problems, expressions, equations and more can be written on cards. For example, later in the same course the technique is used to classify word problems, methods of factoring and linear equations. Cards can include names of likely categories. The information might even be selected by students. For example, to review for a major exam, a group of students might share collected problems, then classify them.

Dittoed sheets may be substituted for index cards in the preparation of information cards. Old or sample tests can be cut up by students and classified. If students cut sheets, be sure to jumble the information so that the order does not reveal assumptions about the categories to be created.
Work format. Again, it is important to distinguish between decisions made for the entire lesson and for the technique used in the core thinking activity part of the lesson. In the model lesson the whole class format is not used with the card sort technique itself. However, it is used in the focusing, skill analysis and metacognitive elements of the thinking lesson.

In the model the students examine the information cards individually as homework. This was done because of time limits during the opening days of school. The higher-order thinking, that is deciding on a classification system and categorizing the information, is done in small groups. Though individual, paired, small group and whole class formats can be used with the technique, it is preferable to use pairs or small groups when determining the categories making up the classification system. If it is necessary to synthesize a classification system for use by the entire class, suggestions for categories should be discussed in small groups before a class consensus is attempted.

Throughout, individual work is assumed if an out of class assignment is given based on the card sort technique. This format is teacher-efficient and might be beneficial, if personalized classification systems are deemed important. However, paired or small group work usually stimulates a variety of ideas and is a motivating factor.

A set of information cards can be prepared for each student or shared. In the model lesson, each student needs
a set of cards for the first night's homework. However, if
class time is available to examine the information, one set
of cards can be divided between a pair or among a small
group of students. When developing the classification
system in class, sharing one set of cards is recommended.
When cards are shared, interaction and interdependence is
increased.

Report of classification system. It is important
that students commit to a classification system. To this
end the technique requires a classification system report.
In the model lesson, each group submits a list of the
categories created and information classified in each. The
classification system which evolves also is reported to the
class in the metacognitive phase of the lesson. Though
beneficial, sharing the classification system is not
necessary to using the card sort technique.

The report can be varied as to mode of communication
and required detail. Reports can be oral, written, drawn
or demonstrated. A simple method of reporting is to show
the arrangement of cards. If category names are not part
of the original set, blank cards are provided for writing
category labels. Students might report by writing their
systems on transparencies, on sheets of newsprint or on
ditto masters.

Categories may be reported with or without all the
information from the cards. The intent of the report is
served by a simple list of categories. In the model lesson, the categories are illustrated with a few examples. In other situations, a complete outline with all categories and items is appropriate.

Students benefit from an opportunity to have their ideas reviewed. Though time may require that written reports be submitted to the teacher, students are then unable to profit from the ideas of others. Also, back and forth interaction during reporting improves understanding and stimulates reflection. This need not happen in the large group format modeled. To allow for more individual interaction, new groups of individuals who did not work together originally can be formed.

In reporting, students should provide at least part of the rationale for the classification system presented. This does not have to be burdensome. If the simple list report is used, the common characteristic that unifies a category can be identified in a phrase after the category label. In some instances, students may be required to report the categories and common features before categorizing information.

**Modification of categories or information.** The classification system may be refined after student reflection. Alterations may be made in the categories selected or in the information used. In the model lesson, students write journal entries recommending modifications
after sharing and discussing systems. In a variation, groups that finish quickly are asked to sequence information within each category or to sequence categories according to priority for review.

Modification might center on the original data. Students can add information not included, but judged important. Also, information which does not fit into any category might be disregarded as not necessary to the purposes at hand. A teacher may set this up by intentionally excluding key data or including irrelevant information. Another modification might require students to create additional cards for each category. For example, problems based on each of the review categories could be created as a modification of the model lesson.

The modification component is regarded as optional. In initial experiences with classification students may need closure. Time limitations may restrict elaboration. However, the vision of mathematics education is better served if students realize classification systems are evolving tools.

Implementation of Targets of the Vision.

The card sort technique incorporates several of the targets for implementing the new vision of mathematics education. Like the other two techniques in this chapter, it is designed to generate mathematical thinking through a student-centered task. The task of comparing and
contrasting information in the process of determining categories develops deeper understanding of the concepts and schemas involved. Students are required to communicate their decision about a classification system. When small groups are used as recommended, students are engaged in communicating about mathematics. Dispositions about mathematics are involved as use of this technique brings into question traditional perceptions. There is more than one 'right' solution; students are actively engaged; and the responsibility for learning shifts from teacher to student. The cards are manipulatives in the unadvertised sense. The card sort technique also provides opportunities for peer and self assessment. Thus, this technique implements many targets of the new vision.

The Equation/Graph Pattern Technique

The equation/graph pattern technique engages students in finding patterns in the relationships between equations and graphs. Computer or calculator graphing allows students to search for patterns rather than bog down in the mechanics of paper and pencil graphing. Though the focus is on the thinking skill of pattern finding, all instructional targets in the framework are integrated.
Purpose of the Technique.

The purpose of the technique is to engage students in identifying patterns. Students are challenged to generalize relationships by graphing an equation, modifying the equation, graphing the modified equation, then observing changes from one graph to the next.

The cognitive skill of finding patterns involves identifying a relationship that occurs repeatedly. It requires observing, hypothesizing and hypothesis testing. Piece by piece analysis of data may be needed in order to make a generalization.

Mathematics is considered a science of patterns. To identify the patterns in the everyday world and represent them mathematically is the essence of what mathematicians do. Furthermore, identification of patterns is key in understanding many mathematical concepts.

A Model Lesson Incorporating the Equation/Graph Pattern Technique.

The lesson modeled is done over three forty minute periods with an Algebra I class. Prior to this lesson, students experience graphing lines by plotting points. It is assumed students are comfortable working with computers, but have little or no experience with a graphing utility.

The format for the model of this technique is descriptive. It is presented as an explanation of the
class to a colleague. The sub-side headings are included to help identify the elements of an effective thinking lesson.

**Focusing.** Breaking the normal routine and meeting in the computer lab usually sparks students' interest before the door is unlocked. As the preassigned pairs move to the machines, I answer the inevitable ‘Can we play Nintendo?’ question and draw a coordinate plane on the board.

Once everyone settles, I draw a line with positive slope and ask “What do you notice?” Responses usually include ‘a line’, ‘a slanted line’, ‘a line slanted up’ and less often ‘a line crossing the positive x axis and the negative y axis’. I repeat drawing and questioning with two or three more lines.

Students are told they will need to identify changes in computer drawn graphs in order to identify a pattern for a group of equations. The computer will draw the graphs of given equations rather than rely on graphs drawn by hand.

**Skill analysis.** I explain that the computer will allow the students to focus on finding patterns. Students are shown a poster-sized sheet of newsprint on which I have written notes which are displayed and saved for use later. The poster is titled ‘Pattern finding’. It includes a definition of the skill, ‘identifying a repeated relationship’ and suggestions for finding patterns. The suggestions include statements such as ‘Be clear on the
type of pattern for which you are searching.', 'Repeat a cycle of observe-hypothesize-predict-test.', 'Generate additional examples when stuck,' and 'Try different perspectives for examining the data.' The list of suggestions is not intended to be all inclusive as the students will revise it at the end of the activity.

After reviewing the definition and list with students, I give needed instruction in use of the graphing utility. Instruction is kept to a bare minimum. Key points and commands are displayed on a transparency which stays available for reference. Students work with their partner at their own pace. Usually one student types and the other records.

Core thinking activity. Each pair is given a single worksheet with detailed directions. The worksheet lists a series of linear equations, such as $y = 2x + 3$, $y = 2x + 5$, $y = 2x + 0$ and $y = 2x - 1$. Students graph the equations, noting how the graphs change. Next students examine the graphs with respect to the equations and list as many observations as possible. Quantity is encouraged.

After examining their list of observations, students circle one that leads to a prediction. For example, 'Graphs of equations with a negative constant cross the y axis below the origin.' Based on the circled hypothesis, students write a specific prediction such as 'The graph of $y = 2x - 6$ will touch the negative portion of the y axis.'
Students then test and evaluate their prediction.

The graph-observe-hypothesize-predict-test cycle is repeated. Students are encouraged to refine conclusions and anticipate the graph before using the computer. While some pairs identify several conclusions, others use the available time to formulate only one. I am flexible, as it is important that all students have the opportunity to complete the cycle successfully.

As students work at the computers, I circulate around the room reassuring students as needed. If students reach a plateau in their exploration, I ask a question to challenge them to investigate more. Near the end of the period, each pair writes a conclusion on the board. Duplication and disagreement are handled in a whole class discussion which I moderate.

For homework, students make two lists. First, they think about the parts of an equation in two variables, then prepare a list of how such equations can be altered. Second, they think about how graphs of lines might differ and make a list of how graphs can be altered. I encourage students to stimulate ideas by looking at equations and graphs in the textbook.

Transfer: core thinking activity repeated. At the start of the second class, students share ideas while I list them on the board under the headings 'Altering Equations' and 'Altering Graphs'. As each suggestion is
made, I write an equation or draw a line and ask students to give an example of the alteration. The list for equations usually focuses on changing coefficients of x and the constants. The graph list usually includes slant, shifts in direction and intersection with the axes. I ask students to use the mathematical terminology for the coefficients, the constants and the intercepts. The term 'slope' is introduced after the equation/graph pattern lesson. If students look at textbook examples, they notice different forms of the linear equation. Though other forms are not suppressed, I indirectly guide students to focus on equations of the slope-intercept form. The summary of homework is done efficiently so students maximize time at the computers.

I remind students that yesterday they investigated how changing the constant in an equation like \( y = 2x + 3 \) altered the graph. Today students repeat the investigation for several different alterations. I review the steps of graphing, observing, hypothesizing, predicting and testing. I explain that the software they are using allows them to alter equations, not graphs. However, they should refer to the 'Altering Graphs' list as they observe changes in the graphs done by the computer.

I also review the finding patterns poster which remains on display. Depending on my observations of the previous class, I may emphasize certain suggestions.
Before working with the computer, pairs receive a worksheet for recording results of the investigation of each series of equations. Sections include 1) what equations are graphed, 2) a statement of how the equation changes, 3) a statement of how the graph changes as a result of changing the equation and 4) a statement of the relationship between the graph and the equation revealed by the pattern of changes. I tell students that I have a few index cards on which are written equation variations, but not a complete set for all pairs. If students want to start using one of the variations on the card, they should see me. However, pairs are encouraged to generate and graph their own altered equations. Most pairs are generating their own equations by the end of the period.

For the remainder of the period, students investigate and record their results. I ask students to leave their worksheet records, so the sheets definitely are available tomorrow. It also affords me the opportunity to review the conclusions.

At the start of the third class, pairs are given five to ten minutes to continue investigation with the computer and to prepare for the sharing of their results with others.

Pairs are combined into groups of four. Each group is asked to submit a summary of all the conclusions that can be supported. If time allows, the groups are challenged to complete statements, such as 'The y-intercept
of the graph is determined by the...in the equation.' and
'The rise or fall from left to right of the graph is
determined by...in the equation.'

Metacognition. About ten minutes before the end of
the class, students comment on how they found patterns.
Remarks include what helped and what did not. I return to
the pattern finding poster and ask students to reflect on
the suggestions. The poster is reworked. Other
suggestions are added.

Instructional Components of the Technique.

In creating a lesson using the equation/graph pattern
technique, teachers make decisions about four instructional
components:

1. graphing tool,
2. type of equation/graph,
3. degree of task open-endedness and
4. pattern report.

The graphing tool refers to the computer software or
graphing calculator used. The type of equation/graph means
what type of function is used. The model concentrates on
simple linear functions. Task open-endedness refers to the
amount of structure in the material presented to students.
Some format for reporting the patterns found must be
designated. Thoughts on use and variations of each
component are discussed in the next section.
Reflection on and Variation of Components.

Graphing tool. Key to this technique is the provision of a mathematical tool to rapidly graph equations. Graphing calculators or computers with graphing utilities are both appropriate. The model lesson opted for computers since graphing calculators are not provided by my school and since students are familiar with computers. Also, available software is user friendly which minimizes time spent on teacher instruction.

An improvement on the lesson makes use of a mathematical tool that will provide an equation, given a graph. Students are able to hypothesize and test predictions about equations as a selection of graphs is plotted. Perceiving relationships in both directions, equation to graph and graph to equation, enhances understanding.

Availability of an appropriate tool and complexity of task determine if equation to graph, graph to equation or both are used. The model lesson focuses on equation to graph to reduce complexity. In a subsequent lesson graph to equation is used for review and extension.

Type of equation/graph. The technique applies to many types of functions. In Algebra I, it also can be used with simple quadratic functions. In Algebra II, it can be used with quadratic functions, exponential functions and logarithmic functions.
Within each type of equation other limitations are possible. The model lesson might or might not include vertical and horizontal lines. Complications can arise by altering the scale of the axes. Introduction of various forms of the same equation need consideration. In the model lesson the slope-intercept form is the main content focus. Later in the course, a lesson based on the equation/graph pattern technique is done which explores the \( ax + by + c = 0 \) form of the linear equation.

Open-endedness of task. The lesson modeled is a compromise between directed discovery and free exploration. To make the task more open-ended, students only receive instructions to explore the effect on the graph of changing one aspect of an equation. Equation selections on worksheets or on cards are not provided. This presents a richer field for conjecture, but students may have difficulty isolating one aspect for change.

Students can be led to specific conclusions by a carefully sequenced set of equation groups for graphing. Presenting a sequence of equations for every pattern to be discovered guarantees coverage of all targeted relationships. However, this approach limits the student generated connections. The model attempts a compromise by decreasing structure from day one to day two.

Pattern report. The model lesson uses worksheets on day one and day two to force students to take structured
notes and to synthesis conclusions as statements. There is less structure as the groups of four complete pattern summaries.

The worksheets are modified or eliminated depending on the students and on other technique variations. For example, if the equations to be graphed are sequenced and presented to students, there is no need to keep track of what is graphed. If students work for an extended time without twenty-four hour breaks, record keeping can be diminished. Some students may need less structured record keeping due to ability or previous training. At the extreme, eliminating all requirements for recording may help students learn the importance of record keeping for themselves. However, if record keeping is eliminated, there still is a need for some means of presenting a summary of conclusions.

Implementation of Targets of the Vision.

The equation/graph pattern technique provides a method of incorporating all eight instructional targets. Schemas are elaborated by building hypotheses from observations. This constructed knowledge is tested by use of a mathematical tool and by discussion with peers. Pattern finding is the focus thinking skill, but classifying, analyzing, observing and comparing are involved. The opportunities to communicate about mathematical ideas and to work in a variety of formats are
abundant. The teacher is freed for on-going assessment and intervention as students take responsibility for the investigation. The use of the computer/calculator is consistent with the perception of a mathematical tool and is a motivating factor which contributes to a positive student attitude. Finally, the task itself allows for active participation by the student, for open-endedness, for student-generated examples that therefore touch the student's reality and for discovery of mathematics.

The Concept Explorations Technique

The concept explorations technique clarifies and elaborates a concept through a series of related tasks or activities. Students examine their current understanding of a concept in an introductory exercise, explore different aspects of the concept through several tasks and report on the resulting concept schema. The key ingredient of this technique is the repeated exploration of a concept through a variety of concrete or common sense representations. Students 'play' with an idea without depending on mathematical abstraction.

Purpose of the Technique.

The purpose of the concept explorations technique is to engage students in conceptualizing. Students begin by examining their existing view of a particular concept.
Next, over several days, students explore aspects of the concept through different activities. A summary activity requires the student to articulate the current understanding of the concept.

Conceptualizing is a complex thinking process that incorporates many cognitive skills. In this sense it is a thinking strategy. It involves classifying and pattern finding, as discussed earlier and other convergent and divergent thinking skills. The complexity of conceptualizing means that the process must take place over time to allow for reflection and refinement.

Developing insight into concept learning is important in studying mathematics. Students need to distinguish the mathematical and non-mathematical perceptions of a concept. Students need to appreciate the constructive nature of concept formation. Students also need to appreciate the network of distinctions and relationships well beyond the textbook definition of a mathematical idea.

A Model Incorporating the Concept Explorations Technique.

In this model, Algebra I students explore the concept of equality. Overall the unit takes a divergent, then a convergent, approach to the concept. An introductory session focuses a discussion on student views of equality and then presents an analysis of the thinking skill of conceptualizing. Over a period of three days small groups of students rotate through six thinking activities or
explorations. Each represents a view or condition of equality. The different ideas and questions the students raise in the explorations are then focused by creating a definition of equality. Metacognition is incorporated into the group discussion of each exploration and into the summary activity.

The model is presented in three parts. The introductory activity, six explorations and the summary task are described. The explanation concentrates on the materials used in each exploration. The structuring of group discussion questions is not detailed.

**Introductory activity.** As a class group students are told that the unit will focus on equality, a key idea in algebra. Questions such as "What is equality?" and "What do people say or think about equality?" are used to generate a list of definitions, assumptions and feelings about equality. As ideas are shared, they are written on the board. Quantity of ideas and involvement of many students are goals at this phase. Comments are not evaluated and the concept of equality is not limited to mathematics.

When ideas are exhausted, the teacher asks "Are there different kinds of equality?" As links emerge, categories are named. At this point distinctions such as political equality, social equality and mathematical equality emerge.
However, any labels achieve the goal of thinking about different perspectives of equality.

Next the teacher explains that through their discussion, students are engaged in conceptualization. A brief, but formal, lecture on the thinking process of conceptualizing is given. The importance of concepts and schemas to the understanding of mathematics is sketched. Distinctions are made between the concept name, the concept characteristics or attributes and examples of the concept. It is explained that the process of conceptualization involves more than memorization of a definition. Conceptualizing involves steps or stages including:

"identify examples...identify common attributes...classify attributes...interrelate categories of attributes...identify additional example/nonexamples...[and]...modify concept attributes/structure" (Beyer 1987, 27). The steps are clarified through examples.

In summary, the teacher asks "In terms of these steps in concept development, what have we done to start conceptualizing equality?" As homework, each student is asked to write a preliminary definition of equality and react to the lecture on conceptualizing.

**Explorations.** Though the specific explorations are different, the use of each is similar. Each is designed to bring out another aspect of the concept of equality. Students work independently of the teacher in groups of
three or four students. Instruction sheets, materials, manipulatives and group exploration report forms are set up at stations. Each group is expected to complete three or four versions of each exploration, but additional tasks are provided as options. At the end of each activity, the small group discusses a series of written questions in completing a group exploration report form. The questions initiate and focus a discussion of the task including how the mathematical content relates to equality, how thinking processes are used and what questions are raised. Students continue metacognitive reflection individually in math journals. Each of the next six sections describes one of the explorations of equality.

**Tangram area outlines.** Students use tangram pieces to fill-in outlines of shapes and figures. The outlines used include geometric shapes, letters of the alphabet and animal figures, each made with the seven tangram pieces. Students are encouraged to play with the pieces and try different possibilities. As an optional task, students create an outline of a familiar object using the seven tangram pieces.

Materials include tangram pieces and outlines to be filled in. As only four sets are needed at one time, plastic tangram pieces were purchased at a local children's store. These easily can be constructed from tagboard. The outlines used include a parallelogram, the letter E and a
The outlines are from a tangram workbook, *Tangramath* (Seymour 1971). Teacher or student constructed outlines can be substituted.

All outlines are of area equal to the sum of the area of the seven tangram pieces. The group exploration report leads students to discuss related ideas, such as, the variety of shapes possible given equal area and shapes that give the illusion of larger or smaller size. Students, also, record strategies that lead to success in filling the outlines.

**Balance weight riddles.** Students examine drawings of a balance scale with platforms holding combinations of cubes, pyramids, and spheres. They are instructed to identify the missing shape needed to balance the platform.

Materials include large index cards on which the balance scales are sketched. The riddles are teacher formulated and produced. To create a riddle, each shape is assigned a numerical value and equal combinations are made. Then only the shapes are drawn on the platforms with a question mark replacing one of the shapes. The 'weight' of objects changes from card to card. Optional riddle cards could be constructed with a mystery shape of uniform weight on both sides of the balance.

In this exploration students work with the concept of equal weight. Furthermore, since students are familiar with the balance scale as a model for solving equations, a
discussion of the equality of algebraic expressions is prompted by the group exploration report. Students often make a comparison between the unknown weight and a variable as well. Strategies for finding the unknown shape are compared and recorded also.

Uncommon unit equations. In this exploration, students justify numerical equations that seem to defy simple arithmetic. For example, $1 - 60 = 23$ does not make sense until students realize one day minus sixty minutes equals twenty-three hours.

Each equation is written on a sheet of paper and laminated. One set of equations is provided to stimulate group discussion. Other equations used include: $5 + 5 = 2$ (fingers, hands), $6 + 24 = 1$ (days, hours, week) and $2 + 12 = 1$ (feet, inches, yard).

Mathematical equations are based on the assumption of common units. The equations presented ignore this principle. Through group discussion and metacognitive journal entries, students formulate and share their opinions of this assumption.

Congruence cuts. Students are asked to cut a figure so that the figure is divided into two parts with the same size and shape. The path of the cut does not have to be a straight line.

Each student receives a set of shapes to divide. Three shapes used are found in Puzzlegams (Pentagram
Additional straight cut puzzles are created by folding an index card and cutting through the double thicknesses. The card is unfolded and the resulting shape is traced.

Equality of size and shape, that is, congruence is explored. Though equality of area and equality of perimeter also apply, these aspects are emphasized in other explorations. This task is important to developing the concept of equality from a geometric perspective. Besides identification of a link with geometry, the group exploration report attempts to have students discuss how visual thinking skills are used throughout mathematics. Students reflect on the use of visual thinking in mathematics in a journal entry.

Four 4’s for equal values. Students are asked to write an expression for the numbers from 1 to 10 using four 4’s (Davidson 1991a). Any operations, grouping symbol or number form (fraction, decimal, exponent, factorial) can be used. For example, 1 = 4.4/4.4. The correct rules for the order of operations must be followed. Students are challenged to create additional expressions equal to the same values or to work on values greater than 10.

The materials for this exploration include only scrap paper plus instructions and the group exploration report sheet. The example given is only one way of writing an
expression for the number. Students are asked to create alternative expressions.

Equality of a variety of numerical representations is explored. This task is repeatedly used as a reference experience when dealing with multiple representations. Students develop strategies for producing numbers, such as, keeping the numbers in position, changing operations or using 4/4 in creating several numbers.

**Geoboard perimeters.** Given a diagram of a figure on a geoboard, students create a figure with equal perimeter, but having a different shape. Students need not state the number of units in the perimeter, but must demonstrate that the perimeters are equal.

Each student is given a geoboard, two rubber bands (a spare in case one breaks) and diagrams of figures. Different figures are used, but a sampling includes a rectangle, a pentomino piece, a diamond and a triangle. At least one figure that goes diagonally between pegs is included. Figures are drawn on dot paper the size of actual geoboards and put in plastic page protectors.

Equality is based on an unknown measure of length. Equality is maintained even though the actual numerical measure remains unknown. The exploration report form asks students to justify that their figures are equal in perimeter to the original figures.
Summary of mathematical content and cognitive process. When all explorations are completed, students remain in their small groups for a summary activity. They discuss their concepts of equality, remaining ambiguities and the process of conceptualization. The groups are asked to prepare a written definition of the concept and to list any remaining questions. Individual journals and group puzzle questions sheets are available for reference.

Each group shares its definition of equality by writing it on the board. Similarities and differences are discussed. One definition is sought through consensus.

Questions are shared and discussed. Though several issues may be resolved, it is hoped that several questions are left unanswered. These can be reexamined as the student’s knowledge grows.

The list of steps in conceptualizing are reexamined. Students are asked to give examples of the steps and react to the procedure. Appropriate modifications are made. Student metacognition is probed to bring out several characteristics of conceptualization. For example, concepts formed are constructed, are abstract, cannot be verified and are usually hierarchial (Seiger-Ehrenberg 1991).
Instructional Components of the Technique.

Four instructional components are key in using the concept explorations technique:

1. concept representations,
2. explorations,
3. timing of explorations and
4. concept summary.

Different representations or aspects of a concept must be identified. For example, equality is considered from both algebraic and geometric perspectives. Explorations or tasks related to the various representations of the concept must be found or created. The sequence and spacing of the exploration presentation must be planned. Also, a format for communicating the refined concept and related schemas needs to be designated.

Reflection on and Variation of Components.

Concept representations. The teacher must identify a concept with characteristics that can be modeled from different perspectives. The different perspectives must be articulated and clear in the teacher's mind. The concept need not be limited to a mathematical perspective. Political and social equality is considered in the introductory activity and some summary definitions. Many mathematical concepts are influenced by non-mathematical connotations, so it is best to have this prior influence exposed.
Explorations. Explorations can be designed using a variety of puzzles, brain teasers, number tricks, alphametics, spatial problems, etc. The technique also encourages the use of a variety of mathematical tools. In the model lesson, tangrams and geoboards are used. Explorations can be designed for other easily available manipulatives. One idea is to explore equality of perimeter, area and volume using Cuisenaire rods (Davidson and Wilcutt 1981). Activities with calculators and computers also can be incorporated. For example, figural logic software might be used (Baker 1988).

Several versions of each exploration are used. An attempt is made to vary the difficulty of the required versions. Students who work quickly or who are intrigued can try additional, more difficult versions. Also, students create their own versions.

Explorations are found in books of mathematical diversions or designed by the teacher. Once the technique is decided on, the teacher can start collecting possible explorations for use with other concepts. For example, for first year algebra, activities could be collected for the concepts of function, variable, infinity and algebraic structure. Catalogs from companies, such as Activity Resources, Creative Publications, Critical Thinking Press & Software, Cuisenaire Company of America, Dale Seymour Publications and the National Council of Teachers of
Mathematics, provide curriculum resources for mathematics classrooms.

When an exploration type is established, students can create other versions. For example, students can create outlines for a tangram area exploration or write expressions for the numbers one to ten using the last four digits of their phone number. In all cases, activities must motivate students to reexamine familiar ideas.

Each exploration involves a group exploration report and metacognitive journal entries. The main purpose of the group exploration report is to stimulate and focus discussion. The structure of the report can vary depending on students' experience with small group work. Though questions on each exploration are different, each form incorporates four types of questions. First, students record solutions, for example, sketch the solutions to the tangram pieces outline or state the mystery values in the balance weight riddles. Second, students discuss and state the aspect of equality represented by the exploration. Next, suggestions and strategies for completing the exploration tasks are listed. Students are queried about what works and what does not work. Finally, any questions emerging from the task or conjectures about equality are recorded.

Journal entries are completed individually. Students often do this writing as homework. Entries vary from reflections on the related mathematics to strategies for
problem solving and conceptualizing. The entries are not 'graded'; rather the teacher responds with comments designed to stimulate further thought.

Timing of explorations. The model presents the concept explorations technique as a short unit done on four consecutive days. Originally it was presented during an odd block of time spanning school terms. One difficulty of this timing is that some students view the unit as 'just games'. The goal of deeper understanding of an abstract mathematical concept should be stressed.

Another approach is to space the explorations over time. Activities can be done in small groups, but all groups do the same task at the same time. Or explorations might be a regular weekly event. The introduction of the particular concept and the lecture on conceptualizing might even occur after the tasks, but before the summary definition. If this sequence is used, students should be allowed additional time to discuss the relationship of the explorations to the concept.

Concept summary. A format for articulating the concluding state of students' conceptualizations must be specified. The written definition of the model forced students to temporary closure, but did not reflect the richness of the concept. Most groups incorporated the different types of equality in their statements, but the format did not highlight the distinctions.
Concept mapping is recommended as an alternative. Simple web or spider maps can be used with minimal student experience. However, the hierarchical nature of many mathematics concepts fits more complex graphic organizers (Clarke 1991).

Implementation of Targets of the Vision.

Deep understanding of concepts and schemas, mathematical thinking and student-centered tasks are clearly implemented in the concept explorations technique. The summary statement requires students to communicate a synthesis of their conceptualization. Communication about mathematics also is enhanced when explorations are done in small groups. Other work formats can be incorporated as well. Explorations selected can take advantage of a variety of mathematical tools. The technique promotes a positive disposition toward mathematics by linking concept construction with exploration disguised as play.

Analysis of Other Models

Three techniques for teaching mathematical thinking were introduced in this chapter. The techniques were analyzed in terms of instructional decisions to be made by the classroom teacher. This approach is not typical of other model lesson presentations which often only present the sequence of activities in the classroom. Furthermore,
alternatives for each instructional decision are considered. Use of the techniques requires teacher reflection and input.

This approach to presenting techniques can be turned into a method for analyzing other models of instruction. Given an exemplary lesson, teachers can analyze the instructional decisions behind the sequential steps. These become the technique components which can then be modified to tailor the model to other situations. Such an analysis will ease transfer to a variety of other situations. The approach allows teachers to view model lessons as independent of particular topics; lessons become examples of a technique which can be applied to many learning situations.

Extrapolating the mode of technique presentation to other situations is an example of how teachers can use the ideas and suggestions of this thesis to spark their own thoughts. Other directions for moving beyond this thesis are made in the next and final chapter.
Overview

This chapter presents two convictions that form the foundation of this thesis. The first is that the processes of learning and doing mathematics should be the focus of restructuring. The second is that teacher reflection leading to ownership is key to successful implementation of the new vision.

The focus on process is discussed in the first section. A distinction is made between the process skills associated with mathematical power and the instructional processes utilized by the classroom teacher. Both deserve priority over an emphasis on content. The definition of mathematical power, the framework of targets, the emphasis on mathematical thinking and the lesson techniques all reflect this attention to process.

The importance of teacher reflection leading to ownership is discussed next. A link is made between teacher ownership and successful implementation of the vision. The assumption of teacher reflection in the framework of targets, the strategy of teaching thinking and the technique presentation are reviewed. In conclusion the need for teacher reflection spirals back to challenging the classroom teacher to understand and implement the vision.
From the definition of mathematical power to the framework of instructional targets to the focus on thinking skills and strategies, this thesis emphasizes process. In planning and implementing the new vision of mathematics education, the processes of doing and of learning mathematics receive attention. However, the symbiosis between content and process is assumed throughout; factual content feeds processes of learning and doing mathematics.

The suggestions for restructuring encompass two types of processes. First are skills and strategies for doing mathematics. These are the process skills that mathematically powerful students acquire. Second, restructuring emphasizes change in the process of instruction. Teachers experienced in the traditional model need to develop a broader range of instructional strategies and techniques. Though these two categories represent different perspectives, they are melded as two sides of the same coin. In both the discussion of goals for students and the analysis of instructional method, process takes precedence over content in attempts to implement the new vision.

**Mathematical Processes as Student Goals.**

Our information society and the evolution of the mathematical sciences have moved us well beyond the point
where even the most advanced student can expect to master the 'facts' of mathematics. Rather, student goals now focus on mathematical skills which can be used to acquire knowledge of any subfield of mathematics and to apply mathematics in one's personal and professional life.

Mathematics is a creative, active process very different from passive mastery of concepts and procedures. Facts, formulas, and information have value only to the extent to which they support effective mathematical activity. Although some fundamental concepts and procedures must be known by all students, instruction should consistently emphasize that to know mathematics is to engage in a quest to understand and communicate, not merely calculate. (Mathematical Sciences Education Board and National Research Council 1990, 12)

The vision of mathematical power represents a shift in emphasis to the processes of doing mathematics. Factual knowledge and procedural algorithms are not dismissed, but become secondary. The traditional content of school mathematics is employed in doing mathematics, but is not viewed as an end in itself. Experienced high school teachers with a traditional background need to redefine student goals to swing the pendulum toward acquisition of process skills.

This conviction is reflected in the framework of targets presented in Chapter II. The targets aim at creating instructional opportunities that develop process skills. Goals include students constructing their own understanding of concepts and schemas, communicating about mathematics, thinking mathematically and displaying a positive disposition toward mathematics. Students also are
expected to work in a variety of formats, to use calculators and computers in extended problem solving and exploration and to engage actively in realistic applications of mathematics. The targets point to student behaviors tied to mathematical skills, not particular content.

The emphasis on process is seen also in recommending a restructuring strategy of teaching thinking. As argued in Chapter III developing mathematical thinking is key in developing mathematical power. The strategy is implemented through the techniques of Chapter IV which are designed to develop the thinking process of classifying, pattern finding and concept formation.

A modification of traditional content topics or their sequence without attention to process skills and instructional method will not help students develop mathematical power as conceived in the new vision. This shift from content to process as the focus of student learning leads to a reexamination of the process of instruction and calls for teachers experienced in the traditional model to expanded their repertoires.

The Instructional Process.

The new vision of mathematics requires fundamental change in the process of instruction. Teachers must rethink the traditional classroom pattern of homework review, new topic lecture and homework drill.
The learning principles summarized in Chapter II represent a synthesis of recent learning research which challenges the traditional model. Emphasized is the construction of knowledge, the influence of prior knowledge, the social aspect of learning and the active nature of learning. Learning is viewed as an active endeavor set in a community of learners.

Thus any attempt at restructuring must expand the teacher’s instructional repertoire beyond the traditional lecture and teacher-led discussion. For example, if students are to communicate about mathematics, they must have opportunities to talk about mathematics with their peers in small groups. Traditional teachers need to develop new techniques to provide this opportunity. It is not the purpose of this chapter to review the specific suggestions for each target, but rather, to point out that a commitment to restructuring mathematics education implies attention to the instructional process. "What students learn is fundamentally connected with HOW they learn it (NCTM 1991, 21)."

The call for teachers experienced in the traditional model to expand their repertoires of techniques is reiterated in the focus in Chapter III on cognitive instruction and the techniques presented in Chapter IV. Lessons that incorporate core thinking activities and metacognition vary from the traditional model and the
responsibility for analyzing, organizing and evaluating knowledge is shared between teacher and student.

To develop the process skills of mathematic power and to expand the instructional process by broadening the range of techniques, teachers need to engage in reflection. Teacher reflection leading to ownership is essential.

Reflection and Ownership

Teacher Ownership.

The teacher is key to opening the door of the classroom to restructuring mathematics education. All the calls for change and the best of the responses do not impact students unless teachers achieve ownership of the vision.

Teacher ownership is not implementation of decisions handed down by authorities. Teachers must be partners in research, planning and evaluation. This is recognized in documents like the NCTM Standards (1989) which allow much leeway in the specifics of implementation. National committees and organizations must continue to view their "role as supporting the efforts of the central person who can bring about meaningful and lasting change: the teacher" (National Research Council 1989, viii).
The Necessity of Reflection.

To achieve ownership, mathematics teachers need to become "reflective practitioners" (Schon 1983). Reflection involves interpretation, examination, analysis and evaluation. Reflection on existing knowledge is embedded in understanding the vision. Reflection in applying the vision to a particular situation happens as implementation strategies and tactics are developed.

Reflecting on the vision. Reflection is needed for teachers to understand the vision. The constructivist view indicates that knowledge is built from the learner's reality and experience. As change agents, teachers become learners who need time to investigate and to synthesize. The framework of instructional targets is presented as a starting point for teachers.

The framework can be altered in many ways. Other targets can be added. Life-long learning skills, connections with other subject areas and classroom climate could all be considered as sections in Chapter II. Targets can be categorized differently. The disposition target could be split into perception of mathematics as a field and perception of mathematics in relation to the individual student. The general targets as outlined in Chapter II might be specified to fit a particular group. For example, targets like reading mathematics, instead of communications or cooperative learning instead of work format.
As soon as teachers have completed a framework of targets, it will need further review. As students change, the framework should be altered. For example, as technology is integrated into elementary and middle school instruction, use of calculators and computers in high school courses will become commonplace. As the teacher's expertise and interests evolve, the framework should evolve. Interdisciplinary curriculum may be a consequence of increased teacher commitment and comfort with learning through major problem solving projects. Neither the elaboration nor the structure of the framework should be etched in stone. Teachers must construct and reconstruct their own versions. Reflection leads to ownership by creating a personal interpretation of the vision.

Reflecting on implementation. Teacher reflection is part of implementing the vision. The new model of instruction presents teachers with a complex, dynamic, interactive task. Teachers need to reflect in applying the vision; there are no preset formulas. "Good teaching demands that teachers reason about pedagogy in professionally defensible ways within the particular contexts of their own work" (NCTM 1991 22).

The strategy and tactics for implementation proposed in this thesis assume modification. In adopting the approach of teaching thinking as the focus for restructuring instruction, teacher reflection and
modification are urged. The strategy obligates teachers to select thinking skills and strategies appropriate for their classes and to modify the depth and breadth of specific presentations to fit student needs. The teacher's metacognitive reflection is also emphasized. The ideal is that teachers develop the mindset of always seeking opportunities to teach mathematical content in the context of developing mathematical thinking.

The presentation of the techniques in Chapter IV assumes teacher reflection for implementation. Though models are included so that the techniques can be viewed in a context, the techniques are not analyzed as a procedural sequence. Rather, the components of the techniques are instructional decisions which must be made by the teacher. The discussion of components presents a range of alternatives for each decision, but implementation requires a teacher to select the alternatives that will best serve a given learning situation. The presentation of technique components and their discussion are designed to promote reflection and ownership.

Ending at the Beginning

Underlying this thesis is the conviction that teacher reflection leading to teacher ownership is necessary to bring the vision of mathematics education into the classroom. The result is that throughout this thesis an
attempt is made to present expert opinion and research in a form that allows classroom teachers to mesh the information with their professional experience and situational requirements. Throughout, an attempt is made to present ideas as take-off points for the practicing teacher.

Understanding and implementation of the new vision are necessary to attain the educational goals delineated by experts. However, the priority of and means to these goals must be charted by classroom teachers. Hopefully, this thesis has provided some direction for experienced teachers as they strive to meet the challenge of restructuring for mathematical power.

The struggle to expand instruction beyond the traditional repertoire is still very much an issue in the day to day teaching of the author. Yet increasingly there are moments when the classroom approximates the descriptions set forth to illustrate the vision. As students are absorbed productively in doing and learning mathematics, they are becoming mathematically powerful. This is the reward for struggling to understand and implement the vision.
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