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Product Bundling: Impacts of Product Heterogeneity and Risk Considerations

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Abstract

Bundling has been extensively studied in the literature and its benefits have been manifested through three perspectives of achieving better price discrimination, helping to save costs, and preserving the power for deterring a potential entrant. In this study, we examine two aspects of bundling which have not been studied before. We examine the impact of product heterogeneity on bundling decisions. We also address risk considerations in a bundling problem. Specifically, we consider a retailer who has the option of selling a bundle of two products (pure bundling policy), or selling the products separately (no-bundling policy). The retailer could also face a product selection problem for which we consider three scenarios of choosing two products with perfectly positively correlated, perfectly negatively correlated or independent reservation prices. We use a Mean-Variance approach to include retailer's risk through her profit variability when maximizing the expected value of profit. We characterize the conditions under which a policy or scenario performs better than the others under the influence of product heterogeneity and/or retailer's risk aversion. Among other findings, we show that optimal bundling price chosen by a risk-averse decision maker cannot be larger than the one chosen by a risk neutral decision maker.

Keywords: Product Bundling, Risk Analysis, Mean-Variance Analysis.

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1. Introduction

Bundling is the sales of two or more separate products in a package (Stremersch and Tellis, 2002), alternatively, it can be viewed similar to volume discount where the volume is based on aggregate sales across products (Nalebuff, 2008). Bundling literature enumerates different reasons for bundling. For instance, it has been shown that a better price discrimination can be achieved, especially when customers' evaluations of products are negatively correlated. Furthermore, bundling can help save transaction or packaging costs. Bundling has also been shown to play as a competitive mechanism by preserving the power for deterring a potential entrant. Of course, there are certain situations in which no-bundling is preferred, either to enhance the profit or to keep distance from legal concerns. Overall, bundling is extensively used in different industries. Bundling of vacation packages, software applications, insurance packages, restaurant menus, consumer products, electronic journals, telecommunication packages, etc. are some of the common applications in daily life related to both manufacturing and service segments. The trend of using bundles is increasing over time due to emergence of offering bundles of services with products, in particular for business segments (Dukart, 2000; Swartz, 2000).

Marketing and economics literature have extensively studied many aspects of product bundling. In this paper we try to analyze two aspects which have not received proper attention in the existing bundling literature, but can have major impacts on bundling decisions.

We first examine the impact of heterogeneity in the two products to be bundled. We look at the heterogeneity from the perspective of customers' reservation prices for the two products. The heterogeneity could be due to the difference in the average prices which customers are willing to pay for each product, e.g. bundling an expensive product with an inexpensive one. For instance, personal computers are sometimes bundled with (low-priced) external audio speakers. As another example, flight tickets are usually bundled with rental cars, where the former could be much more expensive than the latter. For more examples see Brough and Chernev (2012).

The heterogeneity could also be due to the difference in the uncertainty level in the customers' reservation prices for the two products. From the firm's point of view, the uncertainty in the customer reservation prices could be due to firm's lack of information about each customer's valuation of the products. High heterogeneity could happen when an established product is bundled with a new product with high uncertainty in the customers' valuations of the products. For instance, AMC Theaters bundle movie tickets with popcorn and drinks. While customers' valuations of popcorn and drinks are relatively known, their valuations of a new movie are more uncertain. Another example could be the bundle of cell-phone plans and a newly released handset. Customers' valuations of a newly released handset are much more uncertain than customers' valuations of cell-phone plans. This type of heterogeneity might also happen when a new product, whose quality is unknown to customers, is bundled with an established high quality product to signal the quality of the new product (Choi, 2003).

We also examine the impact of firm's risk attitude. To the best of our knowledge, risk considerations have not been studied in the existing bundling literature. In this paper, we use a *Mean-Variance* (MV) approach to examine the impact of risk on bundling decisions. In this approach, the firm maximizes the expected profit while keeps the profit variance below a threshold level. Compared to other risk related parameters, the expected and variance of profit is most readily available to decision makers. Hence, the MV method can be considered as the most practical approach. We will show how the bundling decisions could change when the firm is considering an MV approach (risk-averse) rather than a simple expected profit maximization approach (risk-neutral).

Our model considers a monopolist retailer selling two products to a market whose customers have different valuations for the products. We present a customer's valuation for a product through a reservation price, which indicates the maximum price a customer is willing to pay for it. Hence, the customers' valuation of a product, from the retailer's point of view, is a random variable. In accordance with the majority of bundling studies, we assume uniformly distributed reservation prices. That is, the reservation price of each customer for a product is a draw from a uniform distribution. However, as opposed to most studies, who consider reservation prices normalized between 0 and 1, we consider a general case of any arbitrary range for reservation prices. Although this more general model makes the derivation of results more complicated, it allows us to examine the impact of product heterogeneity on bundling decisions.

The retailer has the choice of applying either pure bundling policy, in which the products are offered only in the form of a bundle and not separately, or no-bundling policy. While customers' reservation prices are independent from each other, the reservation prices of an individual customer for the two different products can be correlated. To capture the impact of this correlation, we present our results for three extreme scenarios: independent, perfectly positively correlated, and perfectly negatively correlated reservation prices. We compare the performance of these scenarios and offer related managerial insights.

The rest of this paper is organized as follows. Section 2 briefly reviews the related literature. In section 3, we describe the model and derive the preliminary results which are used through the rest of the paper, including purchasing probabilities and optimal prices. In section 4, we analyze the impact of product heterogeneity. The impact of risk consideration is presented in section 5. Section 6 closes the paper by our concluding remarks and a few managerial insights. Proofs of all propositions are in Appendix A.

2. Literature review

The literature on the economics of bundling can be categorized into three broad groups: benefits of bundling as a tool for price discrimination (McAfee et al., 1989), as a cost saving mechanism (Evans and Salinger, 2005), and finally as a means of entry deterrence (Carlton and Waldman, 2002; Nalebuff, 2004).

Traditionally, economists have explained bundling as an effective tool for price discrimination since it helps a monopolist to reduce heterogeneity in customer valuations (Bakos and Brynjolfsson, 1999). This means the advantage of bundling is especially apparent when the values of products are negatively correlated. In this case, bundling leads to more homogeneous valuations among customers and thus a greater portion of customer surplus can be captured by the monopolist. The first study on the benefit of bundling from this perspective can be traced back to the influential work of Stigler (1968), followed by structural study of Adams and Yellen (1976), and has continued by other researchers such as Simon and Wübker (1999) and Kühn et al. (2005). These papers mainly explore the primary benefits of bundling in different situations; different from our intention of investigating the impacts of risk considerations and product heterogeneity. Schmalensee (1982) shows that mixed[2] bundling can be profitable for a firm even when customers' valuations are positively correlated as long as the correlation is not near to or equal to one. McAfee et al. (1989) show that even the bundling of independent products can still be better than no-bundling. Moreover, the authors show that if the retailer could monitor the purchases, then a mixed bundling strategy can almost always be more profitable than no-bundling. To achieve this result, the authors assume that the retailer can prevent consumers from

[2] In a mixed bundling strategy the retailer sells the bundle of the products as well as each product separately.

purchasing both product 1 and product 2 separately. As opposed to McAfee et al. (1989), our model focuses only on the case where the retailer cannot monitor the purchases. Instead, we provide insights on the impact of the correlation between the reservation prices, the impact of product heterogeneity, and the role of retailer's risk preferences. We refer interested readers to Kobayashi (2005) for a more detailed review of this literature.

Another theme of studies on bundling has been about transaction cost reduction mostly in the form of bundle discounts (Dewan and Freimer, 2003; Janiszewski and Cunha, 2004; Sheng et al., 2007). In a more recent study, Evans and Salinger (2008) provide a model for the size of discount and highlight the critical role of cost in explaining bundling and tying behavior in comparison with the role of demand in the previous studies. They show that bundling is more profitable when customers are willing to buy all components of the bundle, or when the fixed costs of handling and transaction are high. Their model is based on the assumption that customers' demand for each product is independent of the price (perfectly inelastic demand). In this paper, however, we model customers' demand through their reservation prices for each product. Therefore, demand for each product (or the bundle) depends on the selling price through the probability distribution of the reservation prices. Hence, we can model the impact of heterogeneity in the customers' valuations of the products.

The third advantage of bundling is entry deterrence, which is beyond the scope of this study. The number of such studies is escalating over time (See Whinston, 1990; Carlton and Waldman, 2002; Nalebuff, 2004; Choi and Stefanadis, 2006; Hubbard et al., 2007; Peitz, 2008).

Bundling of information goods is attracting more attention over time due to technological progresses. In fact, bundling of information goods has been a common practice for a while due to cost savings in production and distribution of physical media such as CDs and DVDs. However, benefits of bundling seem to decrease due to significant cost reduction in reproduction and distribution for information goods. Bakos and Brynjolfsson (1999) show that pure bundling of a large number of information goods is still advantageous in special situations, which may never happen in practice. To address this shortcoming, Hitt and Chen (2005) propose the concept of customized bundling: a pricing mechanism whereby customers may select a fixed number of goods out of the total goods available for a fixed price. Such a pricing scheme has different desirable properties due to flexibility and efficiency. Wu et al. (2008) extend the work of Hitt and Chen (2005) and explore the properties of customized bundling using a nonlinear mixed-integer programming approach. All these papers study the bundling of a large number of information goods with very low (or zero) marginal production cost. Our model, however, focuses on the bundling of only two products with arbitrary marginal production costs. We can therefore provide insights on the impact of marginal production cost (see sections 4 and 5) as well as the impact of heterogeneity in customers' valuations of the products. Other researchers who study the bundling of products with zero marginal cost include Ibragimov (2005), Geng, Stinchcombe, and Whinston (2005), and Fang and Norman (2006).

Our base model can be considered as a generalized model of McCardle et al. (2007). Similar to their work, we consider the impact of bundling products on retail merchandising. Our work, however, is different from that study from several aspects. First, we consider only basic products since our objective is to address risk considerations of bundling, not comparing bundles of fashion and basic products. Second, as opposed to that study and most other studies considering normalized reservation prices between 0 and 1, we generalize reservation prices by considering arbitrary upper and lower limits. Specifically, McCardle et al. (2007) considered the range of reservation prices of one product to be a subset of the other one. Our generalized model lets us consider the impact of heterogeneity in customers' valuations of the two products. Another paper which uses a modeling approach similar to ours is Eckalbar (2010). This paper, however, is limited to the case where the lower bound of product reservations is zero. This simplification lets the author provide insights on the mixed bundling. The paper does not address the impact of product heterogeneity or risk consideration.

To be able to focus on the impact of product heterogeneity and risk consideration, in this research, we assume an additive model for the reservation price of the bundle. That is, each customer's reservation price of the bundle is the sum of the customer's reservation prices of the components. This is consistent with the assumptions in Adams and Yellen (1976), Schmalensee (1984), McAfee et al (1989), McCardle et al (2007), and Kramer (2009). Bulut et al (2009) and Venkatesh and Kamakura (2003) provide a model in which the reservation price of the bundle can be superadditive or subadditive. That is, the reservation price of the bundle can be greater or smaller than the sum of the reservation prices of the components. Modeling superadditivity and subadditivity makes the problem formulation considerably more complicated. Therefore, to derive their results, Bulut et al (2009) mostly rely on numerical analysis, while Venkatesh and Kamakura (2003) resort to simplifying assumptions (bundling of products with identical production cost and identical reservation prices). These authors relate the superadditivity and subadditivity of reservation prices to complementarity and substitutability of products, respectively. Although this relation is accepted by many researchers, there are others who provide different perspectives. Popkowski Leszczyc et al (2008) observe subadditivity for complement products. They also show superadditivity for products that are not complement. McCardle et al (2007) relate the complementarity and substitutability of products to the correlation between their reservation prices.

Choi (2003) proposes bundling as a mechanism to sell a new product with unknown quality bundled with an established product with known high quality. The author uses an *informational leverage* approach to show that this bundling could signal the high quality of the new product. Similarly, we consider the impact of the heterogeneity on customers' perception of the two products. Our focus, however, is on the heterogeneity in customer valuation (reservation prices) of the two products.

Heterogeneity in product bundling has been studied in the literature (Adams and Yellen, 1976; Guiltiman, 1987; Tellis, 1986; Stremersch and Tellis, 2002). The focus of these works is on the heterogeneity in the reservation prices of different customer segments. The general conclusion is that heterogeneity in customer segments makes the product bundling more desirable. Our focus in this research, however, is on the heterogeneity of products (not customer segments) in a homogeneous market. For instance, bundling an expensive product with an inexpensive product, or bundling an established product with relatively known demand with a new product for which the customer valuations are more uncertain.

Other aspects of heterogeneity in product bundling are studied through empirical methods. Brough and Chernev (2011) study consumers' perception of the value of a bundle consisting of an expensive product and an inexpensive product. They show that combining expensive and inexpensive items can lead to subtractive rather than additive judgments. Agarwal and Chatterjee (2003) examine the consumers' perceived decision difficulty in selecting from a menu of bundles, where the bundles vary on different attributes including their perceived similarity. They show that similar bundles pose greater choice difficulty than dissimilar bundles. Similar to our paper, Popkowski Leszczyc et al (2008) study the impact of heterogeneity in product uncertainty for high and low value products. Their approach and focus, however, are different from ours. They use an experimental approach to study the impact of heterogeneity on the superadditivity of the reservation prices. The authors conclude that these heterogeneities can change our perception of complement and substitute products. We, on the other hand, use mathematical and numerical modeling to analyze the impact of heterogeneity under different reservation price correlations. For a comprehensive review of bundling literature see Stremersch and Tellis (2002).

In this research, we also examine the impact of firm's risk consideration in bundling decisions. To the best of our knowledge, all the papers on the economics of bundling focus on risk-neutral firms who try to maximize their expected profit. We contribute to this literature by exploring the impact of firm's risk aversion on product bundling decisions. In the literature of the modern theory of risk management, in the absence of decision maker's utility function, variance of profit (as the most practical and readily available risk measure) has been widely employed based on the pioneer work of Markowitz's (1952). In Markowitz's MV approach, a risk-averse decision maker minimizes the risk (i.e., profit variance) while requiring that the expected profit will not fall below a threshold level. Alternatively, as a dual of this model, the risk-averse decision maker can maximize expected profit (reward) as long as the profit variance (risk) is not escalated beyond a threshold level. In this paper, we use the latter approach as it has been used in many different studies (Choi et al, 2008a; Choi et al, 2008b; Martínez-de-Albéniz and Simchi-Levi, 2006). Clearly, in the special case where the profit variance is not a binding constraint (e.g. when the variance threshold level is high enough), the risk-averse decision maker behaves the same as a risk-neutral decision maker whose only objective is maximizing expected profit. In some other studies, a risk-averse decision maker is modeled as a person who tries to maximize the expected profit while penalizing it by a factor (α) of the profit variance (Gan et al, 2011; Wu et al, 2009; Lau, 1980). This approach can be viewed the same as the earlier approach when the variance constraint is binding, in which case α plays the role of a Lagrange multiplier. When α is zero, the objective function becomes the same as a risk-neutral decision maker. Krokhmal et all (2011) and Steinbach and Markowitz (2001) show the equivalency of the three MV approaches. That is, by considering proper values for the expected profit threshold, profit variance threshold, and factor α, the three MV approaches yield the same set of optimal solutions (efficient frontiers).

3. Model Formulation and Preliminary Calculations

A monopolist retailer sells two products *A* and *B* in a homogeneous market whose size is *M* and customers' purchasing behaviors are independent of each other. A customer's valuation of product *i* is represented by his reservation price for that product, r_i, which indicates the maximum price he is willing to pay to buy it. From the retailer's perspective, the customer's reservation price for a product is a random variable whose distribution, we assume, is uniform: $r_i \sim U[l_i, u_i]$ in which $l_i \geq 0,\ i \in \{A, B\}$. It is a common practice in the bundling literature to assume uniformly distributed reservation prices. Our model, however, considers the most general form of uniform distribution, as opposed to most of the existing studies which assume reservation prices normalized between 0 and 1.

We consider two Policies (P): *Pure Bundling* (1) and *No-bundling* (2). In a pure bundling policy only a bundle of two products A and B is offered to the market. This policy is called *pure* bundling since the products are not offered separately along with the bundle. In a no-bundling policy, the products are offered only separately. Note that the policy index ($P = 1$ or 2) is corresponding to the number of pricing decisions the retailer needs to make.

Under each policy, the customers' reservation prices for a given product are assumed to be independent of each other. That is, the valuation of a customer for product *i* is independent of the valuation of another customer for the same product. However, for a given customer, the reservation prices of the two products A and B are not necessarily independent of each other. Therefore, under each policy, we study three extreme *Scenarios* (S): *Independent* (0), *Perfectly Positively Correlated* (+1), and *Perfectly Negatively Correlated* (–1). Under independent scenario, the valuation of a customer for a product is independent of his valuation for the other product. Under the other two scenarios, however, the valuation of a customer for a product determines his valuation for the other product. In these two perfectly correlated scenarios, there is a linear relationship between each customer's reservation prices for the two products:

$$r_B = \begin{cases} l_B + K(r_A - l_A) & if\ S = +1 \\ u_B - K(r_A - l_A) & if\ S = -1 \end{cases}, \tag{1}$$

where $K = b/a,\ a = u_A - l_A,$ and $b = u_B - l_B$. Without loss of generality, we assume $K \leq 1$. That is, we name the product with larger uncertainty in its reservation price as product A.

Perfectly correlated scenarios correspond to the cases where the correlation coefficient between the reservation prices of the two products is either –1 or +1. In this case the value of one random variable identifies the value of the other one. This assumption is consistent with the assumptions in Carbajo et al (1990), Nalebuff (2004), and McCardle et al (2007). In a perfect correlation r_B cannot vary independent of r_A. Nevertheless r_B is a uniformly distributed random variable. Its randomness, however, follows exactly the randomness of r_A. This is the extreme case for the more general case where the correlation coefficient of reservation prices is a number between −1 and +1. In this general case, each random variable can change (to some extent) independent of the other one. Complete independence happens when the correlation coefficient is zero. We choose to present our results only for perfect correlations (and complete independence) to make our mathematical modeling tractable. Appendix C (available as an electronic supplement) shows how the general case of correlation between reservation prices can be modeled. As it can be seen in this appendix, the problem formulation is considerably more complicated. The complexity of the formulation prevents us to analyze the impact of heterogeneity and risk consideration under the general model. Therefore, the results of the paper are presented only for independent and perfectly correlated reservation prices. The numerical results in Appendix C, however, suggests that the changes in the expected and variance of profit are continuous and monotone with respect to the correlation coefficient. Therefore, by providing the results for the special cases of independent and perfectly correlated reservation prices (the two ends and the mid-point of correlation coefficient specturm), we expect to gain general insights about the behavior of the problem when the correlation coefficient changes continuously between −1 and +1.

The retailer sets price p_i for product i in scenario S (−1, 0, or +1) and under policy P (1 or 2). Note that i could be: only AB when $P=1$ and either A or B when $P=2$. A customer may purchase none of the products, or may purchase AB when $P=1$, and purchase either A, B, none, or A+B (separately purchasing both A and B) when $P=2$.

Under no-bundling policy, the marginal cost of each product i is assumed to be $c_i \le u_i$, otherwise, no customer buys product i. Under the pure bundling policy, if $c_{AB} < c_A + c_B$, we say the retailer benefits from economy of bundling. Assuming positive net profit for each item sold and defining $U = u_A + u_B$ and $L = l_A + l_B$, we have the following relations which are being respected throughout the paper:

$$Max(c_i, l_i) \le p_i \le u_i, \quad i \in \{A, B\},$$

$$Max(c_{AB}, L) < p_{AB} < U, \text{ and } \quad Max(c_A, c_B) < c_{AB}. \tag{2}$$

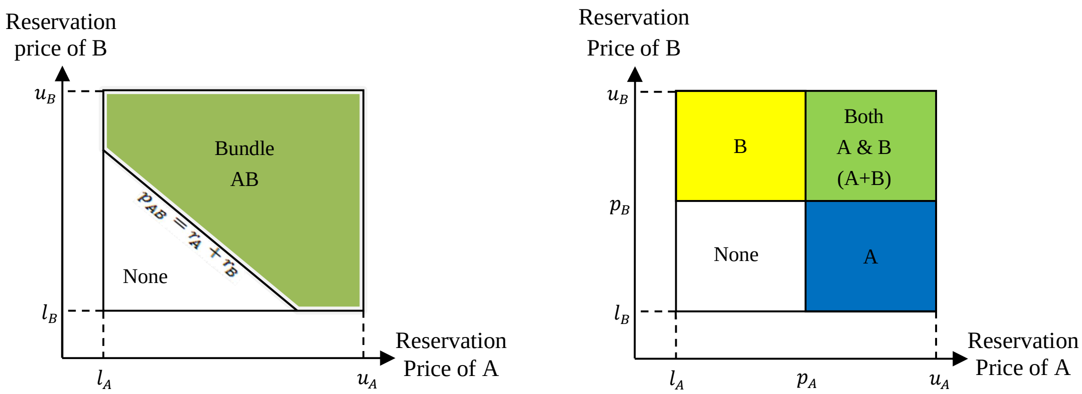
We use π as the total profit earned from each individual customer, and Π as the retailer's total profit. Due to homogeneity of customers and the fact that each customer's purchasing behavior is independent of other customers' purchasing behavior, the expected value and variance of the total profit are, respectively: $E[\Pi] = M.E[\pi]$ and $V[\Pi] = M.V[\pi]$. So, through the rest of the paper, we focus only on the expected and variance of retailer's profit from each individual customer (expected and variance of *total* profit can simply be derived by multiplying by *M*). Through the rest of this section we characterize the purchasing probabilities and the corresponding optimal pricing decisions for each bundling policy.

3.1 Pure Bundling Policy ($P = 1$)

A customer buys the bundle if and only if the bundle price is not more than the sum of his reservation prices for each product individually. Hence, the probability that a customer buys the bundle is: $\Pr(AB) = \Pr(p_{AB} \le r_A + r_B)$. The profit function can then be written as:

$$\pi = \begin{cases} e_{AB} & \textit{with probability of } \Pr(AB) = \Pr(p_{AB} \le r_A + r_B) \\ 0 & \textit{with probability of } 1 - \Pr(AB) \end{cases} \tag{3}$$

where $e_{AB} = p_{AB} - c_{AB}$. Figure 1(a) shows the purchasing behavior of a customer under pure bundling policy. By applying relations in (1), we can characterize the pure bundling purchasing probabilities under each scenario as summarized in table 1.



(a) Pure bundling policy (b) No-bundling policy

Figure 1 - Possible purchasing behavior of customers under the two policies

Scenario	$\Pr(AB)$
$S=-1$	$\begin{cases} 1 & if \quad p_{AB} < u_B + l_A \\ \dfrac{u_A + l_B - p_{AB}}{a-b} & if \quad u_B + l_A \le p_{AB} \le u_A + l_B, K \neq 1 \\ 0 & if \quad u_A + l_B < p_{AB} \end{cases}$
$S=0$	$\begin{cases} 1-\dfrac{(p_{AB}-L)^2}{2ab} & if \quad p_{AB} < u_B + l_A \\ \dfrac{((u_B+l_B)+2(u_A-p_{AB}))}{2a} & if \quad u_B + l_A \le p_{AB} \le u_A + l_B \\ \dfrac{(U-p_{AB})^2}{2ab} & if \quad u_A + l_B < p_{AB} \end{cases}$
$S=+1$	$\dfrac{U-p_{AB}}{a+b}$

Table 1 - Purchasing probabilities of pure bundling policy

The derivation details of the results in tables 1 to 4 can be found in Appendix D (available as an electronic supplement). Using (3), the retailer's expected profit can be derived as follows:

$$E[\pi] = e_{AB}.\Pr(AB) \tag{4}$$

3.2 No-bundling Policy (P=2)

Under no-bundling policy, a customer buys any of the two products if the price of that product is not more than the customer's reservation price for it. Hence the probability that a customer buys product *i* is: $\Pr(i) = \Pr(p_i \le r_i)$. The profit function can then be written as:

$$\pi = \begin{cases} e_A & with\ probability\ of\ \ \Pr(A) = \Pr(p_A \le r_A\ and\ \ p_B > r_B) \\ e_B & with\ probability\ of\ \ \Pr(B) = \Pr(p_B \le r_B\ and\ \ p_A > r_A) \\ e_A + e_B & with\ probability\ of\ \ \Pr(A+B) = \Pr(p_A \le r_A\ and\ \ p_B \le r_B) \\ 0 & with\ probability\ of\ \ 1-\Pr(A)-\Pr(B)-\Pr(A+B) \end{cases} \tag{5}$$

where e_i is the marginal profit of selling product *i*. Figure 1(b) shows the purchasing behavior of a customer under this policy. By applying relations in (1), we can characterize the no-bundling purchasing probabilities under each scenario as summarized in table 2. The purchasing probabilities for product *B*, Pr(*B*), are similar in format to Pr(*A*), but the indices *A* and *B* should be swapped.

Using (5), the retailer's expected profit can be derived as follows:

$$E[\pi] = e_A.\Pr(A) + e_B.\Pr(B) + (e_A + e_B).\Pr(A+B) \tag{6}$$

	$\Pr(A)$	$\Pr(A+B)$
$S=-1$	$\begin{cases} \dfrac{u_A - p_A}{a} & if \quad \dfrac{p_A - l_A}{a} > \dfrac{u_B - p_B}{b} \\ 1 - \dfrac{u_B - p_B}{b} & otherwise \end{cases}$	$\begin{cases} 0 & if \quad \dfrac{p_A - l_A}{a} > \dfrac{u_B - p_B}{b} \\ \dfrac{u_B - p_B}{b} - \dfrac{u_A - p_A}{a} & otherwise \end{cases}$
$S=0$	$\dfrac{(u_A - p_A)(p_B - l_B)}{ab}$	$\dfrac{(u_A - p_A)(u_B - p_B)}{ab}$
$S=+1$	$\begin{cases} \dfrac{p_B - l_B}{b} - \dfrac{p_A - l_A}{a} & if \quad \dfrac{p_A - l_A}{a} < \dfrac{p_B - l_B}{b} \\ 0 & otherwise \end{cases}$	$\begin{cases} 1 - \dfrac{p_B - l_B}{b} & if \quad \dfrac{p_A - l_A}{a} < \dfrac{p_B - l_B}{b} \\ \dfrac{p_A - l_A}{a} & otherwise \end{cases}$

Table 2 - Probabilities of no-bundling policy

Scenario	p^*_{AB}
$S=-1$	$\begin{cases} u_B + l_A & if \quad c_{AB} < u_B + l_A - (a-b) \\ \dfrac{u_A + l_B + c_{AB}}{2} & if \quad u_B + l_A - (a-b) \le c_{AB} \le u_A + l_B, K \ne 1 \\ pure\, bundling\, is\, not\, feasible & if \quad u_A + l_B < c_{AB} \end{cases}$
$S=0$	$\begin{cases} p^*_{AB1} & if \quad c_{AB} < \dfrac{b}{2} - a + u_B + l_A \\ \dfrac{u_B + l_B + 2u_A + 2c_{AB}}{4} & if \quad \dfrac{b}{2} - a + u_B + l_A \le c_{AB} \le u_A + l_B - \dfrac{b}{2} \\ \dfrac{U + 2c_{AB}}{3} & if \quad u_A + l_B - \dfrac{b}{2} < c_{AB} \end{cases}$
$S=+1$	$\begin{cases} \dfrac{U + c_{AB}}{2} & if \quad c_{AB} \ge 2L - U \\ L & otherwise \end{cases}$

Table 3 – Optimal bundling prices for pure bundling policy

Scenario	$E[\pi(p^*_{AB})]$
$S=-1$	$\begin{cases} u_B+l_A-c_{AB} & \text{if} \quad c_{AB}<u_B+l_A-(a-b) \\ \dfrac{(u_A+l_B-c_{AB})^2}{4(a-b)} & \text{if} \quad u_B+l_A-(a-b)\le c_{AB}\le u_A+l_B, K\ne 1 \\ \text{pure bundling is not feasible} & \text{if} \quad u_A+l_B<c_{AB} \end{cases}$
$S=0$	$\begin{cases} (p^*_{AB1}-c_{AB})\left(1-\dfrac{(p^*_{AB1}-L)^2}{2ab}\right) & \text{if} \quad c_{AB}<\dfrac{b}{2}-a+u_B+l_A \\ \dfrac{(u_B+l_B+2u_A-2c_{AB})^2}{16a} & \text{if} \quad \dfrac{b}{2}-a+u_B+l_A\le c_{AB}\le u_A+l_B-\dfrac{b}{2} \\ \dfrac{2(U-c_{AB})^3}{27ab} & \text{if} \quad u_A+l_B-\dfrac{b}{2}<c_{AB} \end{cases}$
$S=+1$	$\begin{cases} \dfrac{(U-c_{AB})^2}{4(a+b)} & \text{if} \quad c_{AB}\ge 2L-U \\ L-c_{AB} & \text{otherwise} \end{cases}$

Table 4 – Optimal expected profit for pure bundling policy

3.3 Optimal Solutions

We can now derive the retailer's optimal solution under different scenarios of pure bundling and then no-bundling policies. Tables 3 and 4 provide the optimal prices and the corresponding maximum expected profits under the pure bundling policy. To make the presentation easier we define p^*_{AB1} as follows.

$$p^*_{AB1}=\frac{2L+c_{AB}+\sqrt{(c_{AB}-L)^2+6ab}}{3} \tag{7}$$

The optimal prices which maximize the retailer's expected profit under no-bundling policy are:

$$p^*_A=\max\left(\frac{c_A+u_A}{2},l_A\right),\quad p^*_B=\max\left(\frac{c_B+u_B}{2},l_B\right), \tag{8}$$

$$E[\pi(p^*_A,p^*_B)]=\begin{cases} ac_{A1}{}^2 & \text{if} \quad c_{A1}\le 1 \\ l_A-c_A & \text{otherwise} \end{cases}+\begin{cases} bc_{B1}{}^2 & \text{if} \quad c_{B1}\le 1 \\ l_B-c_B & \text{otherwise} \end{cases}, \tag{9}$$

where $c_{A1}=\dfrac{u_A-c_A}{2a}$ and $c_{B1}=\dfrac{u_B-c_B}{2b}$.

4. The Impacts of Product Heterogeneity

In this section, we investigate the impacts of relative reservation price uncertainty as well as the relative average customer valuation of the two products on the benefits of product bundling. We use the span of the probability distribution of reservation prices as a measure of its uncertainty, that is *a* and *b* for product *A* and *B*, respectively. A high level of uncertainty could be a result of a high diversity in the retailer's customer base, or it could be due to the retailer's lack of knowledge about the product attractiveness to its potential market. Therefore, $K = b/a$ represent the relative uncertainty in the reservation prices of the two products.

The midpoint of the probability distribution of a reservation price can be considered as a measure for average customers' valuation of the products. That is, $m_A = (l_A + u_A)/2$ and $m_B = (l_B + u_B)/2$. Therefore, we can measure the valuation heterogeneity of the two products with $\eta = m_B / m_A$. To make our comparisons meaningful, while we investigate the impact of η and *K*, we keep the values of $(a+b)$ and $(m_A + m_B)$ constant. The summation $(a+b)$ could, intuitively, represent the total uncertainty in the reservation prices of the two products, while $(m_A + m_B)$ could be a measure of the total average customer valuation of the two products.

4.1. The impact of η

To be able to focus on the impact of η, we assume *K*=1 in this subsection. Without loss of generality we assume $m_B \geq m_A$ $(\eta \geq 1)$.

Proposition 1. *For any given values of a = b, and* $(m_A + m_B)$*, the expected profit of pure bundling policy (all three scenarios) is independent of* η*.*

Proposition 1 states that for any two products with the same relative reservation price uncertainty, the relative customer valuation of the two products, η, does not have any impact on the retailer's expected profit as long as the total valuation of these products is kept constant. This intuitively means that it does not matter whether we bundle a very expensive product with a very cheap product, or we bundle two products with moderate values; both provide the same level of expected profit.

To make the comparison of pure bundling and no-bundling possible, we assume there is no economy of bundling ($c_{AB} = c_A + c_B$) and the cost of each product is proportional to maximum customer valuation of that product. It is easy to verify that under these conditions $c_i = u_i c_{AB} / U$, $i \in \{A, B\}$. Since we have $a = b$, this assumption means that a more expensive product has a higher production cost than the production cost of a cheaper product in a proportional way. This cost structure allows us to focus only on the impact of η.

Proposition 2. *For any given values of $a = b$, and $(m_A + m_B)$, the expected profit of no-bundling policy* (*all three scenarios*) *is increasing in η.*

Propositions 1 and 2 suggest that the value of product bundling (compared to no-bundling policy) decreases as the heterogeneity of the product values increases.

Although demand correlation between the two products does not have any impact on the expected profit under no-bundling policy, the retailer's expected profit from bundled products depends on this correlation (different bundling scenarios). Our results show that for products with very low production costs (information goods for example), bundling of negatively correlated products (S=−1) is more profitable than bundling of positively correlated (S=+1). The profit of bundles of product with independent demands (S=0) is somewhat in between. However, for products with relatively high production costs, or equivalently with low profit margins (commodity products for example), the profit of the positively correlated products is more than the profit of negatively correlated products. The profit of independent products is again somewhere in between.

To better observe how the expected profit of bundling depends on the production cost under different scenarios, we first look at the impact of product bundling on the probability of bundle sales under the three different scenarios. Proposition 3 states this result.

Proposition 3. *For any bundle of two products, the following results hold*

(a) $p_{AB} = \overline{p}_{AB} \quad \Rightarrow \quad \Pr(AB)\big|_{S=-1} = \Pr(AB)\big|_{S=0} = \Pr(AB)\big|_{S=+1} = 50\%$

(b) $p_{AB} < \overline{p}_{AB} \quad \Rightarrow \quad \Pr(AB)\big|_{S=-1} \geq \Pr(AB)\big|_{S=0} \geq \Pr(AB)\big|_{S=+1}$

(c) $p_{AB} > \overline{p}_{AB} \quad \Rightarrow \quad \Pr(AB)\big|_{S=-1} \leq \Pr(AB)\big|_{S=0} \leq \Pr(AB)\big|_{S=+1}$

where $\overline{p}_{AB} = (U + L)/2$.

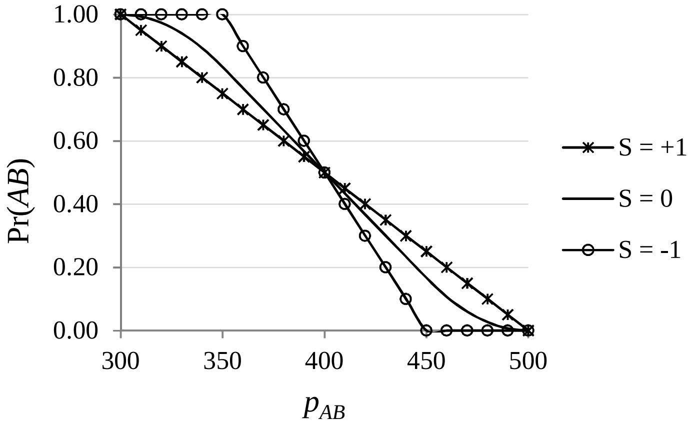


Figure 2 – The behavior of purchasing probabilities across different scenarios of pure bundling ($m_A + m_B = 400; \quad a + b = 200; \quad K = 0.33; \quad \eta = 1$)

Figure 2 depicts the results in proposition 3. We can rewrite equations (4) as $E[\pi]/e_{AB} = \Pr(AB)$. In other words Pr(*AB*) can be considered as the representatives for the expected profit. Moreover, the optimal bundling price is increasing in the bundle cost. Therefore, the behavior of the optimal expected profit vs. bundle cost should be similar to the bundling probability vs. bundle price. Therefore, for very low bundle costs, bundling of negatively correlated products (S=−1) should provide the highest expected profit, while for very high bundle costs, bundling of positively correlated products (S=+1) should provide the highest expected profit. The following proposition proves this result. Let $c_{AB1} = \max\left(U - 2\sqrt{a^2 + b^2}, 0\right)$.

Proposition 4: *For all scenarios, the optimal bundling price and its corresponding expected profit are continuous and decreasing functions of marginal cost of bundling. Moreover,*

(a) when $c_{AB} \le c_{AB1}$ *we have:*

$$E[\pi(p^*_{AB})]_{S=+1} < E[\pi(p^*_{AB})]_{S=0} < E[\pi(p^*_{AB})]_{S=-1} ,$$

(b) when $c_{AB} \ge \overline{p}_{AB}$ *we have:*

$$E[\pi(p^*_{AB})]_{S=+1} > E[\pi(p^*_{AB})]_{S=0} > E[\pi(p^*_{AB})]_{S=-1} .$$

Note that the conditions stated in parts (a) and (b) of proposition 4 are sufficient (not necessary) conditions. Our numerical results show that the results of this proposition can be valid for a much wider range of parameters than what is stated in these conditions. Proposition 4 shows that the behavior of optimal expected profit (under different scenarios) is similar to the behavior of purchasing probabilities. That is, at the lower range of marginal cost of bundling the optimal values of (S=−1) is greater than the optimal values of (S=0) and optimal values of (S=0) is greater than the optimal values of (S=+1). Such a relation is reversed when the marginal cost of bundling is at higher levels. However, as opposed to purchasing probabilities, there is no single marginal cost as a turning point. Instead, there are three different marginal costs of bundling at which different pairs of scenarios have identical optimal values. Figure 3 depicts this behavior for a numerical example.

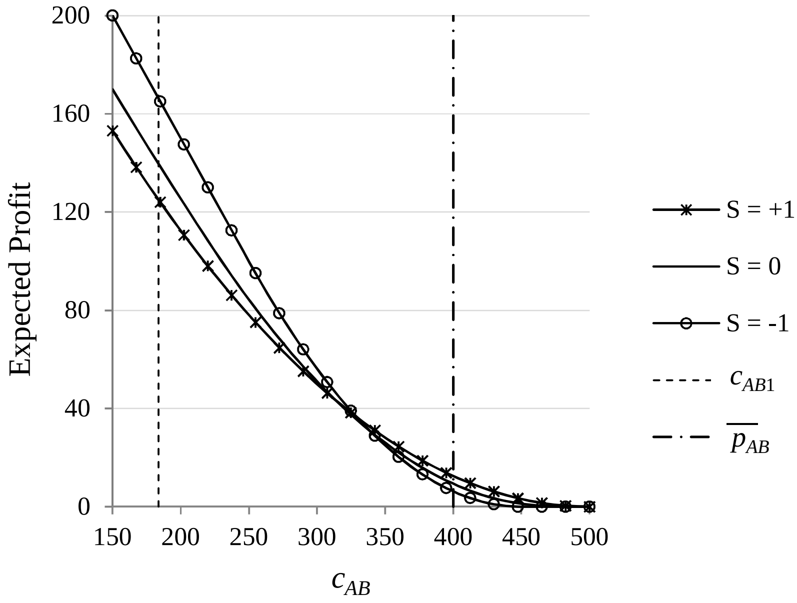


Figure 3 – Expected profit for different scenarios of pure bundling
$(m_A + m_B = 400; \quad a + b = 200; \quad K = 0.33; \quad \eta = 1)$

The result of proposition 5 provides us with the means to compare the profitability of bundling and no-bundling policies under different scenarios and different bundling costs.

Proposition 5. *For any given values of a = b, and* $(m_A + m_B)$, *no-bundling policy is always more profitable than the bundling policy for a bundle of perfectly positively correlated products* (S=+1) *when* $c_{AB} = c_A + c_B$ *and* $c_i = u_i c_{AB} / U$, $i \in \{A, B\}$.

Although we prove proposition 4 for the case of K=1, its result is not limited to this case. McCardle et al (2007) show that no-bundling always performs better than (S=+1) when there is no economy of bundling ($c_{AB} = c_A + c_B$). Their result holds for any value of K as long as $l_B \geq l_A$ and $u_B \leq u_A$. Our numerical results, however, suggest that this result holds even in general ranges of reservation prices. In other words the only way that the pure bundling of positively correlated products can perform better than no-bundling is through a sufficient level of economy of bundling, that is when $c_{AB} < c_A + c_B$. The following proposition states the result for the case where there is economy of bundling.

Proposition 6: *Under perfectly positively correlated scenario, $S=+1$, as longs as optimal bundling prices are greater than the lowest feasible level ($l_A < p_A^*$, $l_B < p_B^*$, $L < p_{AB}^*$), we have:*

$$\frac{(U-c_{AB})^2}{a+b} \geq \frac{(u_A-c_A)^2}{a} + \frac{(u_B-c_B)^2}{b} \quad \Leftrightarrow \quad E[\pi(p_{AB}^*)] \geq E[\pi(p_A^*, p_B^*)]$$

The above relation is a generalized form of the result stated by McCardle et al (2007). Comparing the results of propositions 4 and 5 we can conclude the following corollary.

Corollary 1. *For any given values of $a=b$, and (m_A+m_B) ,while $c_{AB} = c_A + c_B$ and $c_i = u_i c_{AB}/U$, $i \in \{A,B\}$, the following results hold*:

(a) *For very high product costs, no-bundling policy always provides the highest expected profit* (*compared to all scenarios of pure bundling*).

(b) *For very low product costs*

- *For low values of $\eta \geq 1$, no-bundling policy provides lower expected profit compared to bundling of perfectly negatively correlated products* ($S=-1$) *and independent products* ($S=0$).
- *For high values of η* (*if possible*), *no-bundling policy can provide higher expected profit compared to different scenarios of pure bundling*

Figure 4 shows the results through a numerical example.

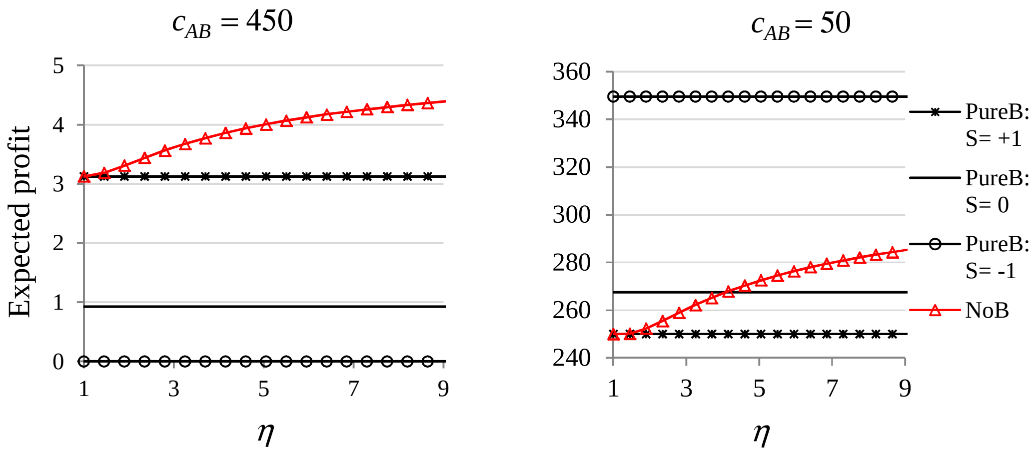


Figure 4 – The expected profit for different policies and scenarios vs. η
$(m_A + m_B = 400; \quad a + b = 200; \quad K = 1.01)$

4.2. The impact of *K*

To investigate the impact of the heterogeneity in the reservation price uncertainty of the two products we look at the impact of changes in K while we keep $\eta=1$ fixed. Similar to previous subsection, to make our comparisons meaningful, we keep the value of $(a+b)$ constant. That is, we compare situations with similar total uncertainty in the reservation prices. The following proposition shows the impact of K on the retailer's expected profit.

Proposition 7. *For any given* $(a+b)$ *and* $\eta=1$, *we have*

(a) The expected profit of pure bundling of perfectly positively correlated products $(S=+1)$ *is independent of K.*

(b) The expected profit of pure bundling of perfectly negatively correlated products $(S=-1)$ *is decreasing in K if* $c_{AB} > U - a$ *and increasing if* $c_{AB} < U - a - (a-b)$.

(c) The expected profit of pure bundling of independent products $(S=0)$ *always lies between the expected profits of* $(S=-1)$ *and* $(S=+1)$ *products.*

(d) The expected profit if no-bundling is always more than or equal to the expected profit of pure bundling of perfectly positively correlated products $(S=+1)$, *when* $c_{AB} = c_A + c_B$ *and* $c_i = u_i c_{AB} / U$, $i \in \{A, B\}$.

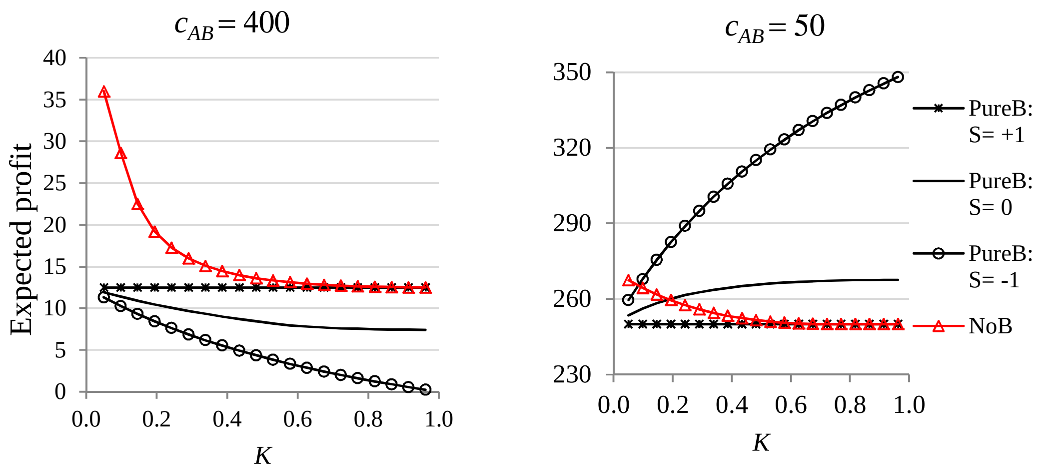


Figure 5 – The expected profit for different policies and scenarios vs. K
$(m_A + m_B = 400; \quad a+b=200; \quad \eta=1)$

Figure 5 demonstrates the results in proposition 6 for a numerical example. We can see that the impact of K is somewhat similar to the impact of η, except that the expected profit of pure bundling depends on the value of K. However, similar to the impact of η, an increase in the heterogeneity level decreases the value of the bundling policies compared to the no-bundling policy. Moreover, the no-bundling policy performs better than all bundling scenarios when the product costs are relatively high. The bundling of negatively correlated products (S=−1) can be higher than the no-bundling policy when the product costs are relatively low, especially when the heterogeneity is not very high. Again, we can show that the bundling of positively correlated products (S=+1) can be more profitable than no-bundling only when there is some economy of bundling. It is also interesting to note that the value of the bundling of negatively correlated products increases with the heterogeneity in the uncertainty of the reservation prices (while keeping the total uncertainty fixed).

5. The Impact of Retailer's Risk Aversion

The results presented in section 4 characterize the optimal parameters for a risk neutral retailer, i.e. for a decision maker who seeks to maximize the expected profit regardless of the involved risk. To characterize the optimal solution for a risk-averse decision maker, we use an MV approach, i.e.;

$$\begin{aligned} &\max\ E[\pi] \\ &subject\ to: V[\pi] \le V_{\max} \end{aligned} \tag{10}$$

where V [.] denotes the variance and V_{max} is the acceptable level of variance (the retailer's risk tolerance). Under this criterion, if the prices which maximize the expected profit result in a profit variance which is smaller than V_{max}, then these prices are optimal. However, if the resulted profit variance is larger than V_{max}, then the retailer should choose a new set of prices which brings down the profit variance to an acceptable level. Profit variance for no-bundling and pure bundling policies can be respectively calculated from equations (11) and (12).

$$V[\pi] = e_{AB}{}^2.\Pr(AB)\big(1-\Pr(AB)\big) \tag{11}$$

$$\begin{aligned} V[\pi] = {} & e_A{}^2 \Pr(A)\big(1-\Pr(A)\big) + e_B{}^2 \Pr(B)\big(1-\Pr(B)\big) + (e_A+e_B)^2 \Pr(A+B)\big(1-\Pr(A+B)\big) \\ & -2\big[e_A e_B \Pr(A)\Pr(B) + e_A(e_A+e_B)\Pr(A)\Pr(A+B) + e_B(e_A+e_B)\Pr(B)\Pr(A+B)\big] \end{aligned} \tag{12}$$

Tables 5 and 6 provide the profit variance for no-bundling and pure bundling policies under different scenarios based on the probabilities calculated in section 3.

Scenario	$V[\pi(p_{AB}^*)]$
$S=-1$	$\begin{cases} 0 & if \quad c_{AB} < u_B + l_A - (a-b) \\ \dfrac{(u_A+l_B-c_{AB})^3(2a-2b-u_A-l_B+c_{AB})}{16(a-b)^2} & if \quad u_B+l_A-(a-b) \le c_{AB} \le u_A+l_B, K \ne 1 \\ pure\,bundling\,is\,not\ feasible & if \quad u_A+l_B < c_{AB} \end{cases}$
$S=0$	$\begin{cases} (p_{AB1}^*-c_{AB})^2\left(1-\dfrac{(p_{AB1}^*-L)^2}{2ab}\right)\dfrac{(p_{AB1}^*-L)^2}{2ab} & if \quad c_{AB} < \dfrac{b}{2}-a+u_B+l_A \\ \dfrac{(u_B+l_B+2u_A-2c_{AB})^3(2u_A-4l_A-u_B-l_B+2c_{AB})}{(16a)^2} & if \quad \dfrac{b}{2}-a+u_B+l_A \le c_{AB} \le u_A+l_B-\dfrac{b}{2} \\ \dfrac{2(U-c_{AB})^3(9ab-2(U-c_{AB})^2)}{9(9ab)^2} & if \quad u_A+l_B-\dfrac{b}{2} < c_{AB} \end{cases}$
$S=+1$	$\begin{cases} \dfrac{(U-c_{AB})^3(U+c_{AB}-2L)}{16(a+b)^2} & if \quad c_{AB} \ge 2L-U \\ 0 & otherwise \end{cases}$

Table 5 – Variance of the optimal expected profit for pure bundling policy

	$V[\pi(p_A^*, p_B^*)]$
$S=-1$	$V_{S=0} - \begin{cases} 0 & \text{if } c_{A1}+c_{B1} \le 1 \\ 2a^2Kc_{A1}c_{B1}\max((1-c_{A1})(1-c_{B1}),0) & \text{otherwise} \end{cases}$
$S=0$	$V_{S=0} = 2a^2\left(c_{A1}{}^3\max(1-c_{A1},0)+K^2c_{B1}{}^3\max(1-c_{B1},0)\right)$
$S=+1$	$V_{S=0} + a^2Kc_{A1}c_{B1}\min(c_{A1},c_{B1})$

Table 6 – Variance of the optimal expected profit for no-bundling policy

5.1. Optimal Prices

Propositions 8 and 9 describe the relation between the optimal price under MV decision criteria, p_i^{MV}, and the optimal price which maximizes the expected profit, p_i^*.

Proposition 8: *Across all scenarios under the pure bundling policy, the unique solution for optimal price,* p_{AB}^{MV}*, under MV decision criteria* (10) *has the following property:*

$$\text{If} \quad V[\pi(p_{AB}^*)] < V_{\max} \quad \text{then} \quad p_{AB}^{MV} = p_{AB}^* \quad \text{else} \quad p_{AB}^{MV} < p_{AB}^*$$

This proposition states that if the bundle price which maximizes the expected profit results in a profit variance larger than the maximum accepted variance, the bundle price should always be lowered to achieve the MV optimal price. This behavior is resulted from the fact that the price maximizing the expected profit is always smaller than the price maximizing the profit variance (see the proof of proposition 8 for details). Similar result holds for the no-bundling policy except for $(S=-1)$.

Proposition 9: *Under no-bundling policy when the scenario is either* $(S=0)$ *or* $(S=+1)$*, the unique solution for optimal prices,* p_A^{MV} *and* p_B^{MV}*, under MV decision criteria* (10) *has the following property:*

$$\text{If} \quad V[\pi(p_A^*, p_B^*)] < V_{\max} \quad \text{then} \quad p_i^{MV} = p_i^* \quad \text{else} \quad p_i^{MV} < p_i^*,\ i \in \{A,B\}$$

The reason that we do not have a similar result for $(S=-1)$ is that, in this scenario, the variance of profit can be decreasing at p_i^*, which means the retailer might need to choose a price higher than p_i^* to bring the profit variance down to the acceptable level. The variance constraint, however, is binding under fewer occasions for $(S=-1)$ scenario, since this scenario has the smallest level of variance compared to the other two scenarios (see proposition 11). Interestingly, the results of propositions 8 and 9 are not limited to uniformly distributed reservation prices. The following proposition states this result.

<u>Proposition 10</u>: *The results of propositions 8 and 9 hold for reservation prices with any probability distribution, as long as the distribution's hazard function is increasing.*

Limiting the distribution to those with an increasing hazard function is a mild condition, since most of the famous probability distributions (Normal, Exponential, Gamma, Poisson, Uniform,…) have this property. Appendix B (available as an electronic supplement) demonstrates this property through a numerical example for a triangular distribution.

5.2. Comparing Bundling Scenarios

To compare the performance of different scenarios of pure bundling policy under MV decision criteria, we define the notion of *dominance* as follows. We say scenario X is dominant over scenario Y if X has equal or higher expected profit and lower profit variance. The dominance of X over Y is shown by $X \searrow Y$. Obviously, $X \searrow Y$ is a sufficient condition to have $CV[\pi]_X < CV[\pi]_Y$, where $CV[\pi]_i$ denotes the coefficient of variation of profit for scenario *i*. The following proposition compares the performance of different scenarios in terms of bundle price.

<u>Proposition 11</u>: *Under pure bundling policy,*

(a) When $p_{AB} < \overline{p}_{AB}$ *we have:* $(S=-1) \searrow (S=0) \searrow (S=+1)$.

(b) When $p_{AB} = \overline{p}_{AB}$ *three scenarios are indifferent.*

(c) When $p_{AB} > \overline{p}_{AB}$ *there is no domination since we have* $V[\pi]_{S=-1} < V[\pi]_{S=0} < V[\pi]_{S=+1}$ *and* $E[\pi]_{S=-1} < E[\pi]_{S=0} < E[\pi]_{S=+1}$.

As we can see, $\overline{p}_{AB}$ is a turning point at which the relative performance of different scenarios changes. Proposition 11 implies that for $p_{AB} < \overline{p}_{AB}$ we have $CV[\pi]_{S=-1} < CV[\pi]_{S=0} < CV[\pi]_{S=+1}$. There is no such a relation for $p_{AB} > \overline{p}_{AB}$ and an MV trade-off (10) should be made.

Exploring the behavior of purchasing probabilities can show us how we have the result stated in proposition 11. From equations (4) and (11), we can see that e_{AB} is the same across different scenarios (for a given value of p_{AB}). Hence, different values of purchasing probabilities are the only reason that we have different values of the expected and variance of profits for different scenarios. In other words, the difference in the performance of scenarios roots in the different behavior of the corresponding purchasing probabilities. We can rewrite equation (11) as $V[\pi]/{e_{AB}}^2 = \Pr(AB)\left(1-\Pr(AB)\right)$. In other words, $\Pr(AB)\left(1-\Pr(AB)\right)$ can be considered as a representative of the profit variance. Figure 6 demonstrates the behavior of this term across different scenarios for different values of bundling price. Although the order of the expected profits of different scenarios turns over when we change bundling price from values smaller than $\overline{p}_{AB}$ to values larger than $\overline{p}_{AB}$ (figure 2), it can be easily verified that for the entire range of possible bundling prices we have: $V[\pi]_{S=-1} < V[\pi]_{S=0} < V[\pi]_{S=+1}$, which intuitively makes sense since for (S=+1) we have the highest correlation of reservation prices and for (S=−1) we have the lowest correlation of reservation prices.

As opposed to pure bundling policy, there is no turning point under no-bundling policy based on the following proposition and corollary.

Proposition 12: *Under no-bundling policy, for any given set of product prices, expected profits are the same across all scenarios and* $V[\pi]_{S=-1} < V[\pi]_{S=0} < V[\pi]_{S=+1}$, *which in turn results in* $CV[\pi]_{S=-1} < CV[\pi]_{S=0} < CV[\pi]_{S=+1}$.

The following corollary is a natural conclusion of proposition 12.

Corollary 2: *Under no-bundling policy, for any set of product prices, we have* $(S=-1) \searrow (S=0) \searrow (S=+1)$.

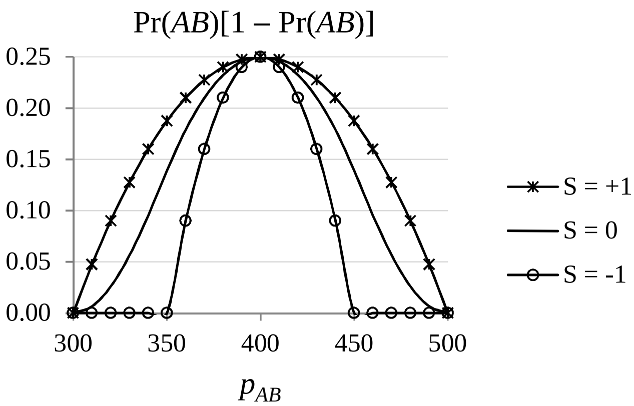


Figure 6 – The behavior of purchasing probabilities across different scenarios of pure bundling
$(m_A + m_B = 400;\quad a+b=200;\quad K=0.33;\quad \eta=1)$

5.3. Bundling vs. No-Bundling

We now try to investigate the conditions under which pure bundling policy is superior to no-bundling policy or vice versa while considering an MV decision criterion. In spite of scenario analysis, comparing different policies leads to more complex relations from which deriving analytical results is not an easy task. Hence, we present these results through numerical analysis. The results presented below is for a case where $a+b=250$ and $m_A+m_B=400$. To consider a full range of possibilities, we also replicated the numerical results for cases where $a+b\in\{100,150,200,250,300\}$. We observed in all these cases the similar behaviors as we observed in the case $a+b=250$ (discussed below). However, for the sake of brevity, we do not present the results for other cases here.

To compare the performance of the two policies, we consider a situation in which the two policies yield equal expected profits (due to economy or diseconomy of bundling). The policy which then provides lower profit variance is more desirable to a risk-averse decision maker. Therefore, we compare $V[\pi(p_{AB}^*)]$ and $V[\pi(p_A^*,p_B^*)]$ for situations where we have $E[\pi(p_{AB}^*)]=E[\pi(p_A^*,p_B^*)]$. To do so, for any pair of (c_A,c_B), we consider the case where the value of c_{AB} is such that it results in $E[\pi(p_{AB}^*)]=E[\pi(p_A^*,p_B^*)]$. When we set the expected profits of the policies equal to each other we can conclude the dominance of the policy by only comparing their variances. Figure 7 plots the variance differences vs. η for pairs of high cost and low cost products. Figure 8 shows the same results for different values of K. We can see a similar pattern between the two sets of figures. That is, regardless of the type of product heterogeneity (K or η), we can observe the following behaviors:

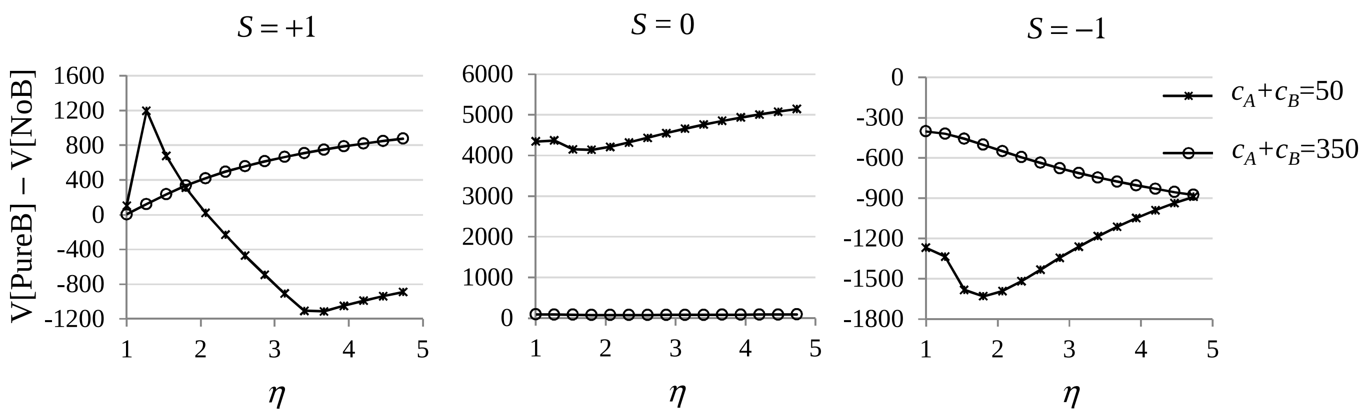


Figure 7 – $V[\pi(p^*_{AB})] - V[\pi(p^*_A, p^*_B)]$ for equal expected profit
$(m_A + m_B = 400; \quad a + b = 250; \quad K = 1.01)$

- For the case of perfectly positively correlated products (S =+1), an increase in the heterogeneity can result in the superiority of the no-bundling policy (lower profit variance for no-bundling) for high cost products. For low cost products pure bundling can be superior when the heterogeneity level is high.
- For the case of independent products (S=0), the no-bundling policy provides lower profit variance and hence is more desirable.
- For the case of perfectly negatively correlated products (S=−1), pure bundling results in lower profit variance for low product costs as the heterogeneity increases. However, for high product costs, the behavior differs with respect to K and η.
    - With respect to η, an increase in heterogeneity of high product costs always results in a lower profit variance of pure bundling which makes it superior to no-bundling.
    - With respect to K, although for low heterogeneity levels the pure bundling has lower profit variance, when the heterogeneity increases, the no-bundling policy becomes more desirable due to its lower profit variance.

We can see that the desirability of the two policies for a risk-averse decision maker could be quite different from that of a risk-neutral decision maker who would be indifferent when the expected profits are equal.

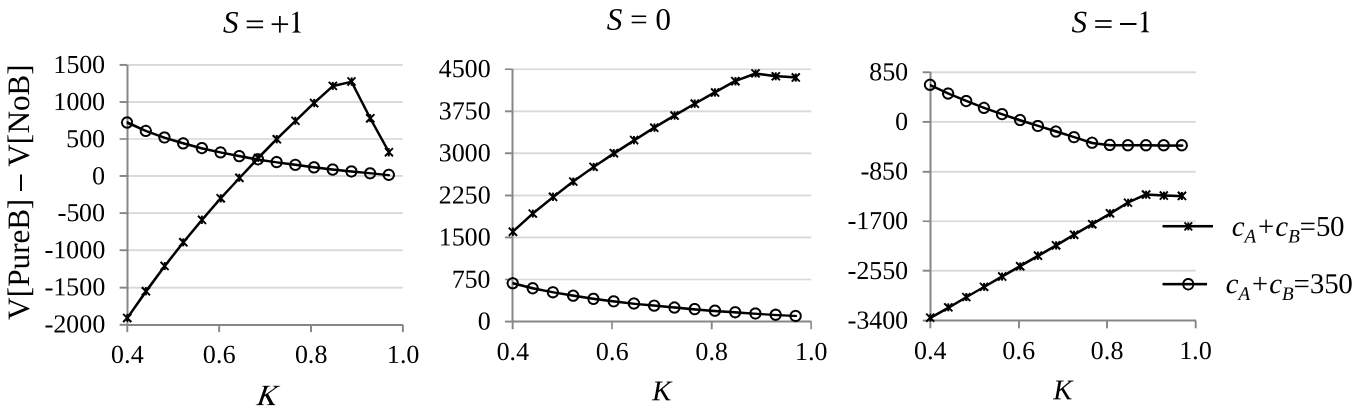


Figure 8 – $V[\pi(p^*_{AB})]-V[\pi(p^*_A, p^*_B)]$ for equal expected profit
$(m_A + m_B = 400; \quad a+b=250; \quad \eta = 1)$

6. Concluding Remarks

In this paper we tried to analyze different aspects of product bundling which have not been received proper attention in the existing literature. We investigated the impact of heterogeneity in the uncertainty level of customers' reservation prices for the two products (the impact of K). High level of heterogeneity could happen when an established product is bundled with a new product with unknown customers' reservation prices. We also investigated the impact of the heterogeneity in the customer valuation of the two products (the impact of η). In this case, the high level of heterogeneity could mean the bundling of an expensive product with an inexpensive one. We analyzed the impact these two types of heterogeneity on the value of product bundling for different scenarios (bundles products with reservation prices that are uncorrelated, perfectly positively, or perfectly negatively correlated). The following managerial insights can be concluded from this analysis (assuming there is no economy of bundling).

- For very high product costs (e.g. commodity products with low profit margins), the expected profit of no-bundling is the highest. The superiority of the no-bundling policy over all scenarios of pure bundling policy increases as the heterogeneity level (in terms of K or η) increases. If bundling is preferable due to other reasons, the bundling of perfectly positively correlated products (S=+1) provides the highest expected profit.
- For very low product costs (e.g. information goods or high end products with large profit margins), when we don't have major heterogeneity, the bundling of the perfectly

negatively correlated products ($S=-1$) provides the highest expected profit. As the heterogeneity level increases, the no-bundling policy performs better and at some point it might outperform the bundling policy.

- The only situation in which the bundling of perfectly positively correlated products can perform better than the no-bundling policy is when there is a certain level of economy of bundling. That is when $c_{AB} < c_A + c_B$.

In addition to the impact of the heterogeneity in the characteristics of the two products, we also explored the impact of retailers' perception of risk. To include retailer's risk aversion in our analysis we use a mean-variance approach, in which the profit variance should not exceed a certain level while the retailer maximizes the expected profit. The followings are the managerial insights which we can learn from this analysis.

- The optimal bundle prices for a risk averse decision maker is always less than or equal to the optimal bundle prices for a risk neutral decision maker.
- For very low bundle prices, the bundling of perfectly negatively correlated products is always the dominant scenario.
- For very high bundle prices, we don't have dominance of a single scenario. An MV trade off should be used to find the scenario which performs better than the others.
- When the bundling and no-bundling policies yields the same expected profit (due to economy or diseconomy of bundling), we have:
    - Bundling of perfectly positively correlatedproducts results in higher profit variances compared to no-bundling policy when the product costs are high (no-bundling is more desirable). For low product costs, however, bundling can have lower profit variances (bundling is more desirable) when the heterogeneity level is high.
    - Bundling of independent products always results in higher profit variances compared to no-bundling policy and hence is less desirable for a risk-averse decision maker.
    - Bundling of perfectly positively and perfectly negatively correlated products behaves the same when the product costs are low.

Our work can be extended from different perspectives. First, in this research we considered only the extreme cases of completely correlated and independent reservation prices for the two products. Considering the full spectrum of correlations could bring new insights to the analysis. Our research can also be extended to consider non-homogenous markets which consist of different segments. Our model was limited to a monopoly environment and considering other market structures such as duopoly or oligopoly could be other extensions.

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Appendix A

Proof of Proposition 1

Let $m_A + m_B = m$. Then it is easy to show that $U = m + (a+b)/2$ and $L = m - (a+b)/2$, which means for fixed values of *m*, *a*, and *b*, the values of *U* and *L* also remain fixed. From the definition of p^*_{AB1}, it is clear that this parameter also remains fixed. We are then able to write the expected profit of the retailer at the optimal price in terms of these fixed parameters. This means that this expected profit is independent of the value of η.

Scenario	$E[\pi(p_{AB}^*)]$
$S=-1$	$\begin{cases} U-a-c_{AB} & if \quad c_{AB} < U-a-(a-b) \\ \dfrac{(U-b-c_{AB})^2}{4(a-b)} & if \quad U-a-(a-b) \le c_{AB} \le U-b, K \ne 1 \\ pure\,bundling\,is\,not\ feasible & if \quad U-b < c_{AB} \end{cases}$
$S=0$	$\begin{cases} (p_{AB1}^*-c_{AB})\left(1-\dfrac{(p_{AB1}^*-L)^2}{2ab}\right) & if \quad c_{AB} < \dfrac{b}{2}-a+U-a \\ \dfrac{(2U-b-2c_{AB})^2}{16a} & if \quad \dfrac{b}{2}-a+U-a \le c_{AB} \le U-b-\dfrac{b}{2} \\ \dfrac{2(U-c_{AB})^3}{27ab} & if \quad u_A+l_B-\dfrac{b}{2} < c_{AB} \end{cases}$
$S=+1$	$\begin{cases} \dfrac{(U-c_{AB})^2}{4(a+b)} & if \quad c_{AB} \ge 2L-U \\ L-c_{AB} & otherwise \end{cases}$

Proof of Proposition 2

Let $m=(m_A+m_B)/2 \;\; and \;\; \delta = m_B - m = m - m_A$.

The following cases are possible (note that $\eta \ge 1 \;\Rightarrow\; c_{A1} \le c_{B1}$)

(a) $c_{A1} \le 2 \;\; and \;\; c_{B1} \le 2$

$$E[\pi(p_A^*, p_B^*)] = \frac{1}{4}ac_{A1}{}^2 + \frac{1}{4}bc_{B1}{}^2 = \frac{1}{4a}(u_A-c_A)^2 + \frac{1}{4b}(u_B-c_B)^2$$

$$E[\pi(p_A^*, p_B^*)] = \frac{1}{4a}u_A{}^2(1-\frac{c_{AB}}{U})^2 + \frac{1}{4b}u_B{}^2(1-\frac{c_{AB}}{U})^2 = \frac{1}{4a}\left(u_A{}^2 - u_B{}^2\right)(1-\frac{c_{AB}}{U})^2$$

$$E[\pi(p_A^*, p_B^*)] = \frac{1}{4a}\left((m-\delta+a/2)^2 + (m+\delta+a/2)^2\right)(1-\frac{c_{AB}}{U})^2$$

$$E[\pi(p_A^*, p_B^*)] = \frac{1}{4a}\left((m+a/2)^2 + \delta^2\right)(1-\frac{c_{AB}}{U})^2 \quad \Rightarrow \quad \frac{\partial E[\pi(p_A^*, p_B^*)]}{\partial \delta} \ge 0$$

(b) $c_{A1} \le 2 \;\; and \;\; c_{B1} > 2$

$$E[\pi(p_A^*, p_B^*)] = \frac{1}{4}ac_{A1}{}^2 + (l_B - c_B) = \frac{1}{4a}(u_A-c_A)^2 + (u_B - b - c_B)$$

$$E[\pi(p_A^*, p_B^*)] = \frac{1}{4a} u_A{}^2 (1-\frac{c_{AB}}{U})^2 + u_B(1-\frac{c_{AB}}{U}) - b$$

$$E[\pi(p_A^*, p_B^*)] = \frac{1}{4a}\left(m-\delta+a/2\right)^2 (1-\frac{c_{AB}}{U})^2 + (m+\delta+a/2)(1-\frac{c_{AB}}{U}) - b$$

$$\frac{\partial E[\pi(p_A^*, p_B^*)]}{\partial \delta} = \frac{-1}{2a}\left(m-\delta+a/2\right)(1-\frac{c_{AB}}{U})^2 + (1-\frac{c_{AB}}{U})$$

$$\frac{\partial E[\pi(p_A^*, p_B^*)]}{\partial \delta} = \frac{1}{2}(1-\frac{c_{AB}}{U})\left[2-\frac{1}{a}\left(m-\delta+a/2\right)(1-\frac{c_{AB}}{U})\right] = \frac{1}{2}(1-\frac{c_{AB}}{U})\left[2-\frac{1}{a}u_A(1-\frac{c_{AB}}{U})\right]$$

$$\frac{\partial E[\pi(p_A^*, p_B^*)]}{\partial \delta} = \frac{1}{2}(1-\frac{c_{AB}}{U})\left[2-\frac{1}{a}\left(u_A - c_A\right)\right] = \frac{1}{2}(1-\frac{c_{AB}}{U})\left[2-c_{A1}\right] \geq 0$$

(c) $c_{A1} > 2 \;\; and \;\; c_{B1} > 2$

$$E[\pi(p_A^*, p_B^*)] = (l_A - c_A) + (l_B - c_B) = (u_A - a - c_A) + (u_B - b - c_B)$$

$$E[\pi(p_A^*, p_B^*)] = u_A(1-\frac{c_{AB}}{U}) - a + u_B(1-\frac{c_{AB}}{U}) - b = (u_A + u_B)(1-\frac{c_{AB}}{U}) - 2a$$

$$E[\pi(p_A^*, p_B^*)] = (2m+a)(1-\frac{c_{AB}}{U}) - 2a \quad \Rightarrow \quad \frac{\partial E[\pi(p_A^*, p_B^*)]}{\partial \delta} = 0$$

<u>Proof of proposition 3</u>

Given the fact that $u_B + l_A \leq \overline{p}_{AB} \leq u_A + l_B$, using probability relations of table 1, the proposition 3 can be proved. In the rest, we prove the proposition for only $p_{AB} < \overline{p}_{AB}$ and it can be similarly proved for $p_{AB} > \overline{p}_{AB}$. Note that Pr(*AB*)=50% across all scenarios at $p_{AB} = \overline{p}_{AB}$ around which probabilities are linearly decreasing with different slopes ($\frac{1}{a-b} < \frac{1}{2a} < \frac{1}{a+b}$ respectively corresponding to *S*=–1, *S*=0, and *S*=+1). This means, when $p_{AB} < \overline{p}_{AB}$, we have: $\frac{u_A + l_B - p_{AB}}{a-b} > \frac{((u_B + l_B) + 2(u_A - p_{AB}))}{2a} > \frac{U - p_{AB}}{a+b}$. Furthermore, when $p_{AB} < u_B + l_A$, we have: $1 > 1 - \frac{(p_{AB} - L)^2}{2ab}$. It is easy to verify that we also have: $1 - \frac{(p_{AB} - L)^2}{2ab} > \frac{U - p_{AB}}{a+b}$ (after simplifying: $p_{AB} - L < \frac{2ab}{a+b} < \frac{2b}{1+K} < b$ which is true as $p_{AB} < u_B + l_A$). Thus, $p_{AB} < \overline{p}_{AB} \Leftrightarrow \Pr(AB)\big|_{S=-1} > \Pr(AB)\big|_{S=0} > \Pr(AB)\big|_{S=+1}$ as illustrated in Figure 2.

Proof of proposition 4

It is easy to verify that p^*_{AB} is a continuous and increasing function of c_{AB}. By considering relations amongst bundling cost limits of table 3 we have:

$$2L - U < \frac{b}{2} - 1 + u_B + l_A < \min\left(u_B + l_A - (a-b), u_A + l_B - \frac{b}{2}\right)$$
$$< \max\left(u_B + l_A - (a-b), u_A + l_B - \frac{b}{2}\right) < u_A + l_B$$

It is easy to verify that optimal prices of the three scenarios have the following unique intersections, as illustrated in figure A1:

(a) $c_{AB} \geq c_{AB1} = \frac{b}{2} - a + u_B + l_A \Leftrightarrow p^*_{AB}\big|_{S=0} \geq p^*_{AB}\big|_{S=-1}$

(b) $c_{AB} \geq c_{AB2} = u_B + l_A - a \Leftrightarrow p^*_{AB}\big|_{S=+1} \geq p^*_{AB}\big|_{S=-1}$

(c) If $\sqrt{a^2+b^2} \leq U/2$ then $c_{AB} \geq c_{AB3} = U - 2\sqrt{a^2+b^2} \Leftrightarrow p^*_{AB}\big|_{S=+1} \geq p^*_{AB}\big|_{S=0}$.

Otherwise $p^*_{AB}\big|_{S=+1} < p^*_{AB}\big|_{S=0}, \forall c_{AB}$.

By considering the facts that $c_{AB1} = c_{AB2} + \frac{b}{2}$ (or $c_{AB1} > c_{AB2}$) and when c_{AB3} exists then $c_{AB2} > c_{AB3}$ (or $c_{AB1} > c_{AB2} > c_{AB3}$), one can conclude the when $\sqrt{a^2+b^2} \leq U/2$, for $c_{AB} < c_{AB3}$ we have: $p^*_{AB}\big|_{S=+1} < p^*_{AB}\big|_{S=0} < p^*_{AB}\big|_{S=1}$ (note that p^*_{AB1} at its lowest value ($c_{AB} = 0$) is greater than L), and when $c_{AB} > c_{AB1}$ we have: $p^*_{AB}\big|_{S=+1} > p^*_{AB}\big|_{S=0} > p^*_{AB}\big|_{S=-1}$.

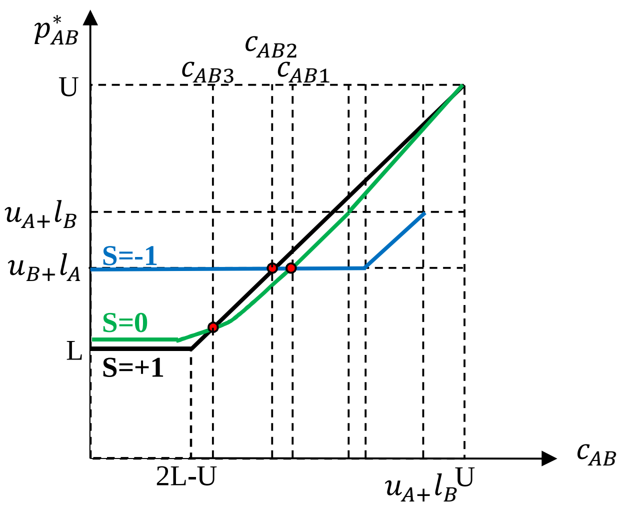

Figure A1 – Intersections of optimal bundling prices of the three scenarios of pure policy

Relations of optimal bundling prices are direct results of the above relation between optimal bundling prices. For expected profits, based on results of proposition 3 and the relations of p_{AB}^*, and the fact that $c_{AB1} < \overline{p}_{AB}$ for (a) we have: $p_{AB}^*\big|_{S=+1} > p_{AB}^*\big|_{S=0} > p_{AB}^*\big|_{S=-1}$ and $\Pr(AB)\big|_{S=-1} \le \Pr(AB)\big|_{S=0} \le \Pr(AB)\big|_{S=+1}$ (similarly, for (b)). Thus, optimal expected profits also follow the same relations as optimal bundling prices.

<u>Proof of proposition 5</u>

We proved that the expected profit of pure bundling is independent of η for any given $a = b$, and $(m_A + m_B)$, while the expected profit of no-bundling is increasing in η. Therefore, it is enough to prove that these two expected profits (S=+1) are equal at $\eta = 1$.

$$\left.\begin{array}{l}\eta = 1 \\ a = b\end{array}\right\} \Rightarrow \begin{cases} u_A = u_B = U/2 \\ l_B = l_B = L/2 \\ c_A = c_B = c_{AB}/2 \end{cases}$$

We consider two possible cases:

(a) $c_{AB} \ge 2L - U \Leftrightarrow \dfrac{U - c_{AB}}{U - L} \le 2 \Leftrightarrow \dfrac{2u_A - 2c_A}{2u_A - 2l_A} \le 2 \Leftrightarrow \dfrac{u_A - c_A}{2a} \le 1 \Leftrightarrow c_{A1} \le 1$

Similarly, $c_{AB} \ge 2L - U \Leftrightarrow c_{B1} \le 1$

$$E[\pi(p_{AB}^*)] = \frac{(U - c_{AB})^2}{4(a+b)} = \frac{(U - c_{AB})^2}{8a}$$

$$E[\pi(p_A^*, p_B^*)] = ac_{A1}{}^2 + bc_{B1}{}^2 = \frac{(u_A - c_A)^2}{2a} = \frac{(U/2 - c_{AB}/2)^2}{2a} = \frac{(U - c_{AB})^2}{8a}$$

(b) $c_{AB} < 2L - U \Leftrightarrow c_{A1} < 1 \;\; and \;\; c_{B1} < 1$

$$E[\pi(p_A^*, p_B^*)] = (l_A - c_A) + (l_B - c_B) = L - c_{AB} = E[\pi(p_{AB}^*)]$$

<u>Proof of proposition 6</u>

The proposition can be easily proved by substituting relation (9) and relation of S=+1 in table 4.

Proof of Proposition 7

Let $m = (m_A + m_B)/2$ *and* $\delta = (a-b)/2$. Since $\eta = 1$, the value of *m* is fixed and δ is half of the difference between the spans of the two distributions. As before $(a+b)$ and *U* are fixed. It would be easy to verify that $a = (a+b)/2 + \delta$, $b = (a+b)/2 - \delta$, and $\delta = [(1-K)/(1+K)][(a+b)/2]$.

Proof of part (a): It is obvious from table 4.

Proof of part (b):

$$c_{AB} < u_B + l_A - (a-b) = U - a - (a-b) \;\Rightarrow\; E[\pi(p^*_{AB})] = U - \frac{a+b}{2} - \delta - c_{AB}$$

The right hand side is decreasing in δ. Since δ is decreasing in *K*, the expected profit is increasing in *K*.

$$c_{AB} \geq u_B + l_A - (a-b) = U - a - (a-b) \;\Rightarrow\; E[\pi(p^*_{AB})] = \frac{\left(U - \frac{a+b}{2} + \delta - c_{AB}\right)^2}{4(2\delta)}$$

$$\frac{\partial E[\pi(p^*_{AB})]}{\partial \delta} = \left(\frac{U - \frac{a+b}{2} + \delta - c_{AB}}{8\delta^2}\right)\left(U - a - c_{AB}\right)$$

The right hand side is positive only if $c_{AB} \geq U - a$. Therefore, for $c_{AB} \geq U - a$, the expected profit is increasing in δ. Since *K* is decreasing in δ, the expected profit is decreasing in *K* if $c_{AB} \geq U - a$.

Proof of part (c): We can conclude it from proposition 4.

Proof of part (d): We first prove that the expected profit of no-bundling is decreasing in *K*. First consider the case where $c_{A1} \leq 1$ *and* $c_{B1} > 1$. Therefore,

$$E[\pi(p^*_A, p^*_B)] = a{c_{A1}}^2 + b{c_{B1}}^2 = \frac{1}{4}\left(1 - \frac{c_{AB}}{U}\right)^2 \left[\frac{{u_A}^2}{a} + \frac{{u_B}^2}{b}\right]$$

We know that $u_A = m + a/2$ and $u_B = m + b/2$. Also, $a = (a+b)/(1+K)$ and $b = (a+b)K/(1+K)$. Therefore,

$$E[\pi(p_A^*, p_B^*)] = \frac{1}{4}\left(1 - \frac{c_{AB}}{U}\right)^2 \left[\frac{1+K}{a+b}\left(m + \frac{a+b}{2}\frac{1}{1+K}\right)^2 + \frac{1+K}{(a+b)K}\left(m + \frac{a+b}{2}\frac{K}{1+K}\right)^2\right]$$

Let $A = \frac{1+K}{a+b}\left(m + \frac{a+b}{2}\frac{1}{1+K}\right)^2 \quad and \quad B = \frac{1+K}{(a+b)K}\left(m + \frac{a+b}{2}\frac{K}{1+K}\right)^2$.

$$\frac{\partial A}{\partial K} = \frac{1}{a+b}\left(m^2 + \frac{1}{4}\left[\frac{a+b}{1+K}\right]^2\right) \quad and \quad \frac{\partial B}{\partial K} = \frac{-1}{(a+b)K^2}\left(m^2 + \frac{1}{4}\left[\frac{(a+b)K}{1+K}\right]^2\right).$$

$$\frac{\partial(A+B)}{\partial K} = \frac{1}{a+b}\left\{m^2 - \frac{1}{K^2}m^2\right\} \le 0 .$$

Therefore, the expected profit of no-bundling is decreasing in *K*. Since the expected profit of no-bundling and the expected profit of bundling of two perfectly positively correlated products (S=+1) is equal at *K*= 1 and $\eta = 1$, we can conclude that the expected profit of no-bundling is always greater than the expected profit of (*S*= +1) for all *K*<1.

Proof of proposition 8 and 10.

Here we prove a general case for pure bundling, not limited to uniform distribution. Let $f_{AB}(.)$ and $F_{AB}(.)$ be the probability density function and cumulative distribution function of r_{AB}, respectively. Then, we have: $E[\pi] = (p_{AB} - c_{AB})(1 - F_{AB}(p_{AB}))$, and $V[\pi] = (p_{AB} - c_{AB})^2 F_{AB}(p_{AB})(1 - F_{AB}(p_{AB})) = E[\pi](p_{AB} - c_{AB})F_{AB}(p_{AB})$.

$$\frac{\partial E[\pi]}{\partial p_{AB}} = (1 - F_{AB}(p_{AB})) - (p_{AB} - c_{AB}) f_{AB}(p_{AB}) = 0 \ \Rightarrow \ p_{AB}^* = c_{AB} + \frac{1 - F_{AB}(p_{AB}^*)}{f_{AB}(p_{AB}^*)}$$

$$E[\pi(p_{AB}^*)] = (p_{AB}^* - c_{AB})^2 f_{AB}(p_{AB}^*)$$

We can prove that p_{AB}^* is the unique maximizer of the expected profit as long as r_{AB} has an increasing hazard function since $p_{AB}^* - c_{AB} = \frac{1 - F_{AB}(p_{AB}^*)}{f_{AB}(p_{AB}^*)}$. The left hand side of the above equation is an increasing function of p_{AB}^* with a negative *y*-intercept. The right hand side of this equation is the inverse of *hazard function*. Since the hazard function is increasing, its inverse is a decreasing function. The right hand side of the equation has a positive *y*-intercept. As a result, this equation has a unique solution, p_{AB}^*. To calculate the CDF of r_{AB} in terms of the distribution functions of r_A and r_B, we look at correlation of products under each scenario. For the variance of profit we have:

$$\frac{\partial V[\pi]}{\partial p_{AB}} = \frac{\partial E[\pi]}{\partial p_{AB}}(p_{AB} - c_{AB})F_{AB}(p_{AB}) + E[\pi]F_{AB}(p_{AB}) + E[\pi](p_{AB} - c_{AB}) f_{AB}(p_{AB})$$

<u>S=0:</u>

$$F_{AB}(p_{AB}) = \Pr(r_{AB} \le p_{AB}) = \Pr(r_A + r_B \le p_{AB}) = \int_0^{p_{AB}} \Pr(r_A \le p_{AB} - y) f_B(y)dy$$
$$= \int_0^{p_{AB}} F_A(p_{AB} - y) f_B(y)dy$$

<u>S=+1:</u> Using relation (1), we have:

$$F_{AB}(p_{AB}) = \Pr(r_{AB} \le p_{AB}) = \Pr(r_A + r_B \le p_{AB}) = \Pr(r_A + l_B + K(r_A - l_A) \le p_{AB})$$
$$= \Pr\left(r_A \le \frac{p_{AB} - l_B + Kl_A}{1+K} \right) = F_A\left(\frac{p_{AB} - l_B + Kl_A}{1+K} \right)$$

<u>S=−1:</u> Using relation (1), we have:

$$F_{AB}(p_{AB}) = \Pr(r_{AB} \le p_{AB}) = \Pr(r_A + r_B \le p_{AB}) = \Pr(r_A + u_B - K(r_A - l_A) \le p_{AB})$$
$$= \Pr\left(r_A \le \frac{p_{AB} - u_B - Kl_A}{1-K} \right) = F_A\left(\frac{p_{AB} - l_B - Kl_A}{1-K} \right)$$

Across all scenarios, $\frac{\partial V[\pi]}{\partial p_{AB}} > 0$ for any $p_{AB} < p^*_{AB}$. Thus, $p_{AB} < p^*_{AB} \Rightarrow \frac{\partial E[\pi]}{\partial p_{AB}} \ge 0 \Rightarrow \frac{\partial V[\pi]}{\partial p_{AB}} > 0$.

<u>Proof of proposition 9 and 10.</u>

We also prove a general case for no-bundling, not limited to uniform distribution. Let $\alpha = \Pr(A) + \Pr(A+B)$, $\beta = \Pr(B) + \Pr(A+B)$ and $\gamma = \Pr(A+B)$ then

$$E[\pi] = e_A\alpha + e_B\beta = E[\pi_A] + E[\pi_B] .$$

Similar to the proof of proposition 8 and 11, in case of general distribution we have: $p_i^* = c_i + \frac{1-F_i(p_i^*)}{f_i(p_i^*)}$ and $E[\pi(p_i^*)] = (p_i^* - c_i)^2 f_{AB}(p_i^*)$. Now, we look at each scenario separately for product A. Similar results hold for product B.

<u>S=0:</u> In this case, $\gamma = \alpha\beta$. So:

$$\frac{\partial V[\pi]}{\partial p_A} = \frac{\partial E[\pi_A]}{\partial p_A} e_A F_A(p_A) + E[\pi_A]F_A(p_A) + E[\pi_A]e_A f_A(p_A) = \omega$$

<u>S=+1:</u> In this case, $\gamma = \alpha$ if $\alpha < \beta$ else $\gamma = \beta$. So,

$$\frac{\partial V[\pi]}{\partial p_A}=\begin{cases}\omega+2\dfrac{\partial E[\pi_A]}{\partial p_A}e_B F_B(p_B) & if \quad \alpha\geq\beta \\ \omega+2E[\pi_B]\left(F_A(p_A)+e_A f_A(p_A)\right) & otherwise\end{cases}$$

So, when $S\neq -1$, $\frac{\partial V[\pi]}{\partial p_A}>0$ for any $p_A<p_A^*$. Thus, $p_A<p_A^*\Rightarrow\frac{\partial E[\pi]}{\partial p_A}\geq 0\Rightarrow\frac{\partial V[\pi]}{\partial p_A}>0$.

Proof of proposition 11.

Since $E[\pi]$ and $V[\pi]$ have respectively common multiplier of e_{AB} and $e_{AB}{}^2$ across all scenarios, we should only focus on probabilistic terms, i.e., Pr(*AB*) for $E[\pi]$ and *V*1=Pr(*AB*).(1−Pr(*AB*)) for $V[\pi]$. Pr(*AB*) behaves according to the proposition 3, across scenarios. However, $\frac{\partial V1}{\partial \Pr(AB)}=1-2\Pr(AB)$ which is negative (positive) when $p_{AB}<\overline{p}_{AB}$ ($p_{AB}>\overline{p}_{AB}$). Considering proposition 3, we can then conclude that at any bundling price we always have: $V1|_{S=-1}\leq V1|_{S=0}\leq V1|_{S=+1}$ (equality happens only at $p_{AB}=\overline{p}_{AB}$, see figure 6). Thus, when $p_{AB}<\overline{p}_{AB}$, the notion of dominance exists since expected profit and variance behave adversely (i.e., $E[\pi]|_{S=-1}\leq E[\pi]|_{S=0}\leq E[\pi]|_{S=+1}$ and $V[\pi]|_{S=+1}\leq V[\pi]|_{S=0}\leq V[\pi]|_{S=-1}$). When $p_{AB}>\overline{p}_{AB}$, notion of dominance does not exists since expected profit and variance behave similarly (i.e., $E[\pi]|_{S=-1}\leq E[\pi]|_{S=0}\leq E[\pi]|_{S=+1}$ and $V[\pi]|_{S=-1}\leq V[\pi]|_{S=0}\leq V[\pi]|_{S=+1}$). Given the fact that at special case of $p_{AB}=\overline{p}_{AB}$ expected profits and variances across scenarios are the same (respectively, $\frac{e_{AB}}{2}$ and $\frac{e_{AB}{}^2}{4}$), the three scenarios are indifferent. This concludes proof of proposition 11.

Proof of proposition 12.

Across all scenarios, we have the following relations:

$$\alpha=\Pr(A)+\Pr(A+B)=\frac{u_A-p_A}{a} \quad \text{and} \quad \beta=\Pr(B)+\Pr(A+B)=\frac{u_B-p_B}{b} \quad \text{and} \quad \gamma=\Pr(A+B)$$

Teherefore:

$$\gamma=\begin{cases}\min(\alpha,\beta) & if \quad S=+1 \\ \alpha\beta & if \quad S=0 \\ \max(\alpha+\beta-1,0) & if \quad S=-1\end{cases}$$

So, we car rewrite (5) as:

$$\pi = \begin{cases} e_A & with\ probability \quad \alpha - \gamma \\ e_B & with\ probability \quad \beta - \gamma \\ e_A + e_B & with\ probability \quad \gamma \\ 0 & with\ probability \quad 1 - \alpha - \beta + \gamma \\ \max(\alpha + \beta - 1, 0) & if \quad S = -1 \end{cases}$$

Therefore, we have: $E[\pi] = e_A\alpha + e_B\beta = E[\pi_A] + E[\pi_B]$, which is the same for all scenarios.

$$\begin{aligned} V[\pi] &= e_A{}^2\alpha(1-\alpha) + e_B{}^2\beta(1-\beta) + 2e_Ae_B(\gamma - \alpha\beta) \\ &= E[\pi_A]e_AF_A(p_A) + E[\pi_B]e_BF_B(p_B) + 2e_Ae_B(\gamma - \alpha\beta) \end{aligned}$$

As intuitively expected, it is easy to verify $V[\pi]\big|_{S=-1} \le V[\pi]\big|_{S=0} \le V[\pi]\big|_{S=+1}$.